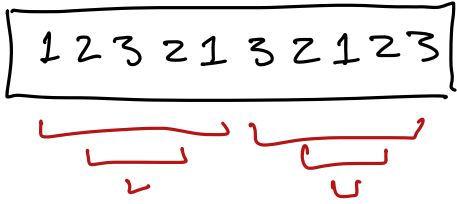


Midterm 1 - next Monday 7-9pm



- No lecture Thu - review session
- HW 0, 1, 2, 3
- 4 questions
- One handwritten cheat sheet
- Conflict exam Tue

Gregory Rawlins - Nuts and Bolts

Given n nuts and n matching bolts different sizes

Task: match every bolt to corr. nut.

Only operation:

- Compare one bolt to one nut
- bolt bigger
 - nut bigger
 - match

Equivalent: sort nuts and/or bolts by size

Brute force: $\Theta(n^2)$ worst case

Mergesort (☹) Quicksort! Deterministic: $\Theta(n^2)$ worst case

Random pivot: $O(n \log n)$ expected

Partitioning nuts and bolts: $2n-1$ tests

Pick bolt
Partition nuts
n tests
Partition bolts
$n-1$ tests

$$\begin{aligned}
 E[T(n)] &= 2n-1 + E_k [T(k-1) + T(n-k)] \\
 &= 2n-1 + \sum_{k=1}^n \frac{1}{n} (E[T(k-1)] + E[T(n-k)]) \\
 &= 2n-1 + \frac{2}{n} \sum_{k=0}^{n-1} E[T(k)]
 \end{aligned}$$

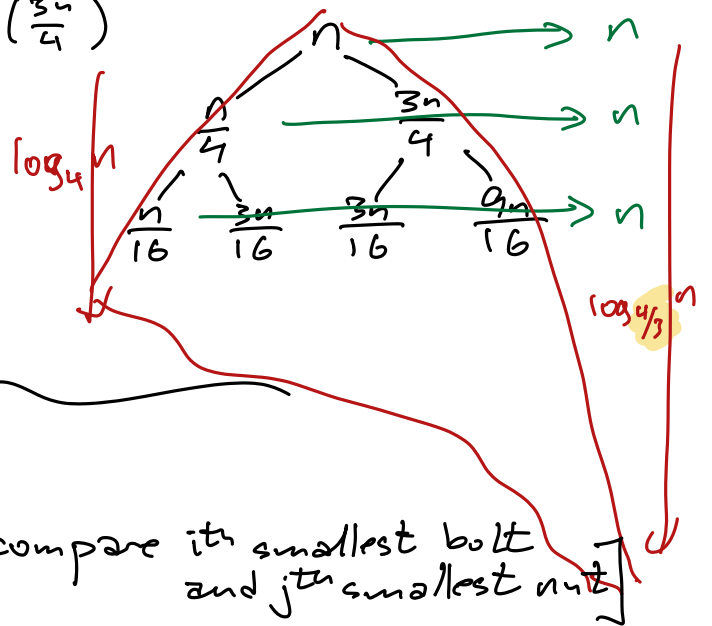
Pivot is good if $\frac{n}{4} < k < \frac{3n}{4}$

$\Pr[\text{good}] = 1/2$

$$\begin{aligned}
 \bar{T}(n) &= \frac{1}{2} E[T(n) | \text{good}] + \frac{1}{2} E[T(n) | \text{bad}] \\
 &\leq 2n-1 + \frac{1}{2} (\bar{T}(\frac{n}{4}) + \bar{T}(\frac{3n}{4})) + \frac{1}{2} \bar{T}(n)
 \end{aligned}$$

$$T(n) \leq 4n - 2 + T\left(\frac{n}{4}\right) + T\left(\frac{3n}{4}\right)$$

$$= O(n \log n) \checkmark$$



Sum of indicator variables

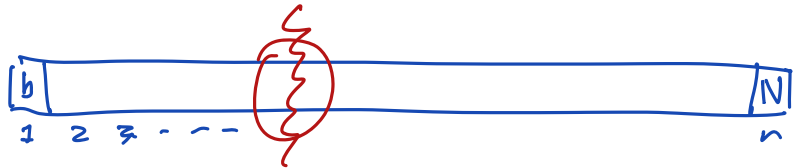
$$T(n) = \sum_{i,j} X_{ij} \quad X_{ij} = \left[\begin{array}{l} \text{compare } i^{\text{th}} \text{ smallest bolt} \\ \text{and } j^{\text{th}} \text{ smallest nut} \end{array} \right]$$

$$E[T(n)] = \sum_{i,j} E[X_{ij}] \quad \text{linearity of } E!$$

$$= \sum_{i,j} (\Pr[X_{ij}=1] + 0 \cdot \Pr[X_{ij}=0])$$

$$\Pr[X_{7,7} = 1] = 1$$

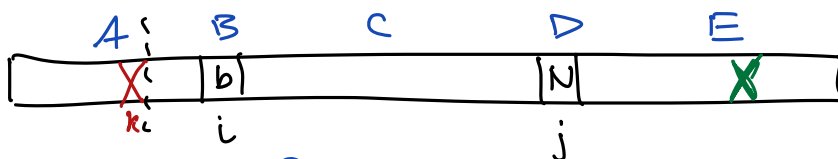
$$\Pr[X_{1,n} = 1] = \frac{2}{n}$$



Case 1: b_j is pivot $\Rightarrow X_{1n} = 1$

Case 2: b_n is pivot $\Rightarrow X_{1n} = 1$

Case 3: $1 < k < n \Rightarrow X_{1n} = 0$



$$\Pr[X_{ij} = 1] = \begin{cases} \text{recurse} & k < i \\ 0 & i = k \\ 1 & i < k < j \\ \text{recurse} & k = j \\ & k > j \end{cases}$$

$k < i$
 $i = k$
 $i < k < j$
 $k = j$
 $k > j$



2 \leftarrow # cases where we compare $i:j$
 $|j-i|+1$ \leftarrow cases where we either compare $i:j$
 or split i and j

$$\Pr[X_{ij}=1] = \frac{2}{|j-i|+1} \text{ unless } i=j$$

We compare bolt i and nut j iff the first pivot chosen from $i..j$ is either i or j

$$E[T(n)] = \sum_{i,j} \Pr[X_{ij}=1]$$

$$= n + 2 \sum_{i=1}^n \sum_{j=i+1}^n \frac{2}{j-i+1}$$



$$4n \ln n - 7n + 4H_n$$

