CS/Math 473 – Algorithms

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Please ask questions!!

What is this?

Algorithms for CS undergrads + grads or nearby

Resources

Policy stuff

Groupwork
Academic integrity

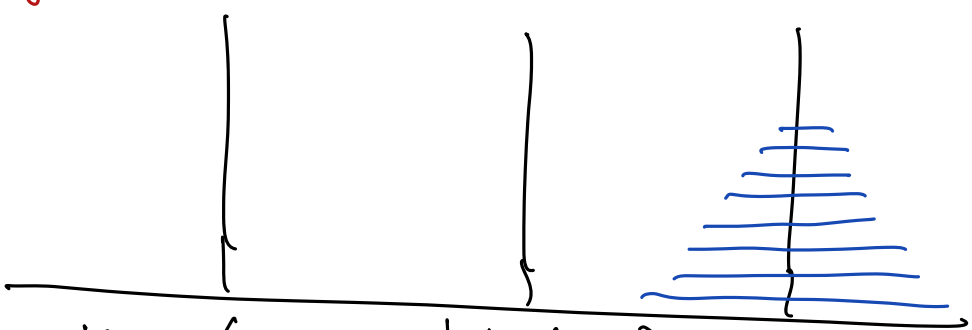
Don't be a jerk.

COVID etc. DIKES

Algorithms

Recursion

Reduce to smaller instances + delegate
—or—
just do it



Tower of Hanoi

- move one disk at a time
- never put larger disk on top of a smaller disk

```
Hanoi(n, src, dst, tmp):
  if n > 0:
    Hanoi(n-1, src, tmp, dst)
    move disk n
    Hanoi(n-1, tmp, dst, src)
```

n : int ≥ 0

check boundary assumptions

Believe in Recursion Fairy

∅ is your friend

Running time = # moves

$$T(n) = \begin{cases} 0 & \text{if } n=0 \\ T(n-1)+1+T(n-1) & \text{otherwise} \end{cases}$$

$T(n) = 2^n - 1$

n	0	1	2	3	4	5	...
$T(n)$	0	1	3	7	15	31	...

Theorem: $T(n) = 2^n - 1$ for all $n \geq 0$.

Proof: Let n be an arbitrary integer ≥ 0

Assume for all $k < n$ that $T(k) = 2^k - 1$.

Two cases:

$n=0$: $T(0) = 0 = 2^0 - 1$ ✓

$n>0$: $T(n) = 2T(n-1) + 1$
$= 2(2^{n-1} - 1) + 1$ [IH]
$= 2^n - 1$ ✓

Thus, $T(n) = 2^n - 1$.

Integer Multiplication

```
  123
 x456
 ----
  738
 615
492
-----
56088
```

$O(n^2)$ time

Kolmogorov '53 "n^2 conjecture"

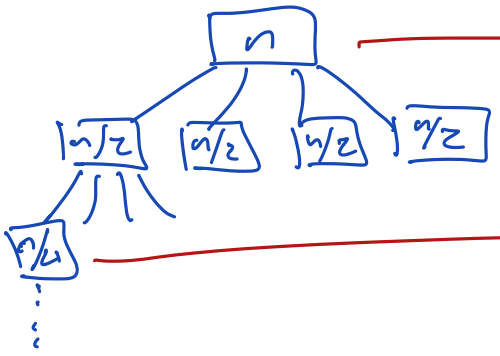
Karatsuba

$$x = a \cdot 10^{n/2} + b$$
$$y = c \cdot 10^{n/2} + d$$

(x, y: n digits)

$$x \cdot y = ac \cdot 10^n + (ad + bc)10^{n/2} + bd$$

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$



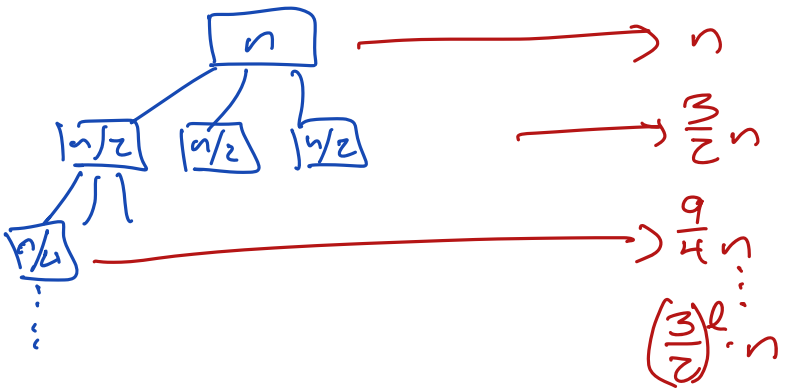
$$\sum_{l=0}^{L} 2^l \cdot n = n \sum_{l=0}^{L} 2^l$$
$$= n O(2^L) = O(n^2)$$

$L = \log_2 n$

$$(a-b)(c-d) = ac - ad - bc + bd$$

$$ad + bc = ac + bd - (a-b)(c-d)$$

$$T'(n) = 3T'(n/2) + O(n)$$



$$T(n) = O\left(n \cdot \left(\frac{3}{2}\right)^{\log_2 n}\right)$$

$$= O\left(n \cdot 3^{\log_2 n}\right)$$

$$= O\left(n \cdot n^{\log_2 3/2}\right)$$

$$= O(n^{1.6\ldots})$$

2019

$O(n \log n)$ time

$$A^{\log_B C} = C^{\log_B A} = e^{\frac{\ln A \ln C}{\ln B}}$$

$$S(n) = O(n) + S(n/2)$$

$O(n)$