CS 473 ✧ Fall 2022

Homework 9

Due **Thursday**, November 17, 2022 at 9pm

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1. Every year, Professor Dumbledore assigns the instructors at Hogwarts to various faculty committees. There are n faculty members and c committees. Each committee member has submitted a list of their *prices* for serving on each committee; each price could be positive, negative, zero, or even infinite. For example, Professor Snape might declare that he would serve on the Student Recruiting Committee for 1000 Galleons, that he would *pay* 10000 Galleons to serve on the Defense Against the Dark Arts Course Revision Committee, and that he would not serve on the Muggle Relations committee for any price.

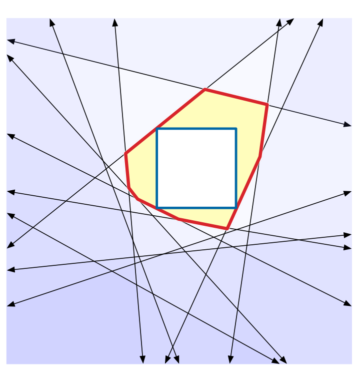
   Conversely, Dumbledore knows how many instructors are needed for each committee, and he has compiled a list of instructors who would be suitable members for each committee. (For example: "Dark Arts Revision: 5 members, anyone but Snape.") If Dumbledore assigns an instructor to a committee, he must pay that instructor's price from the Hogwarts treasury.

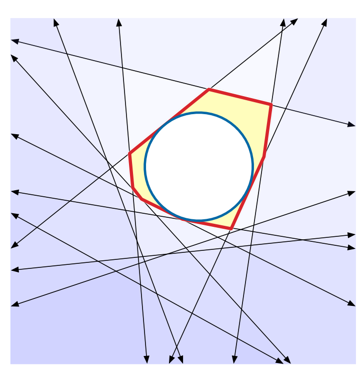
   Dumbledore needs to assign instructors to committees so that (1) each committee is full, (2) only suitable and willing instructors are assigned to each committee, (3) no instructor is assigned to more than three committees, and (4) the total cost of the assignment is as small as possible. Describe and analyze an efficient algorithm that either solves Dumbledore's problem, or correctly reports that there is no valid assignment whose total cost is finite.

2. Suppose we are given a sequence of n linear inequalities of the form $a_i x + b_i y \le c_i$. Collectively, these n inequalities describe a convex polygon P in the plane.

   (a) Describe a linear program whose solution describes the largest square with horizontal and vertical sides that lies entirely inside P.

   (b) Describe a linear program whose solution describes the largest circle that lies entirely inside P.





3. Alex and Bo are playing ***Undercut***. Each player puts their right hand behind their back and raises some number of fingers; then both players reveal their right hands simultaneously. Thus, each player independently chooses an integer from 0 to 5.[1] If the two numbers do not differ by 1, each player adds their own number to their score. However, if the two numbers differ by 1, then the player with the lower number adds *both* numbers to their score, and the other player gets nothing. Both players want to maximize their score and minimize their opponent's score.

   Because Alex and Bo only care about the *difference* between their scores, we can reformulate the problem as follows. If Alex chooses the number i and Bo chooses the number j, then Alex gets M_{ij} points, where M is the following 6×6 matrix:

$$M = \begin{pmatrix} 0 & 1 & -2 & -3 & -4 & -5 \\ -1 & 0 & 3 & -2 & -3 & -4 \\ 2 & -3 & 0 & 5 & -2 & -3 \\ 3 & 2 & -5 & 0 & 7 & -2 \\ 4 & 3 & 2 & -7 & 0 & 9 \\ 5 & 4 & 3 & 2 & -9 & 0 \end{pmatrix}$$

   (In this formulation, Bo's score is always zero.) Alex wants to maximize Alex's score; Bo wants to minimize it.

   Neither player has a good *deterministic* strategy; for example, if Alex always plays 4, then Bo should always play 3. Exhausted from trying to out-double-think each other,[2] they both decide to adopt *randomized* strategies. These strategies can be described by two vectors $a = (a_0, a_1, a_2, a_3, a_4, a_5)^\top$ and $b = (b_0, b_1, b_2, b_3, b_4, b_5)^\top$, where a_i is the probability that Alex chooses i, and b_j is the probability that Bo chooses j. Because Alex and Bo's random choices are independent, Alex's expected score is $a^T M b = \sum_{i=0}^{5} \sum_{j=0}^{5} a_i M_{ij} b_j$.

   (a) Suppose Bo somehow learns Alex's strategy vector a. Describe a linear program whose solution is Bo's best possible strategy vector.

   (b) What is the dual of your linear program from part (b)?

   (c) So what *is* Bo's optimal strategy, as a function of the vector a? And what is Alex's resulting expected score? (You should be able to answer this part even without answering parts (a) and (b).)

   (d) Now suppose that Alex knows that Bo will discover Alex's strategy vector before they actually start playing. Describe a linear program whose solution is Alex's best possible strategy vector.

   (e) What is the dual of your linear program from part (d)?

   (f) **Extra credit:** So what *is* Alex's optimal Undercut strategy, if Alex knows that Bo will know that strategy?

   (g) **Extra credit:** If Bo knows that Alex is going to use their optimal strategy from part (f), what is Bo's optimal Undercut strategy?

   Please express your answers to parts (a)–(e) in terms of arbitrary $n \times n$ payoff matrices M, instead of this specific example. You may find a computer helpful for parts (f) and (g).

[1] In Hofstadter's original game, players were not allowed to choose 0 for some reason.

[2] "They were both poisoned. I've spent the last several years building up an immunity to iocaine powder."