cs473: Algorithms

Lecture 4: Dynamic Programming

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University of Illinois at Urbana-Champaign

September 4, 2019

Overview

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logistics:

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- pset1 out,

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- recognizing that dynamic programming applies to a problem can be non-obvious

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- dynamic programming on trees often generalizes to graphs that have low *treewidth*

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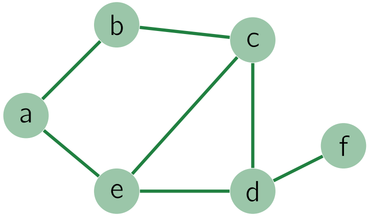

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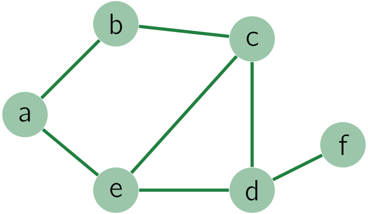

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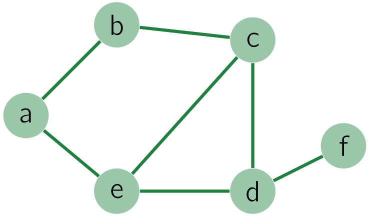

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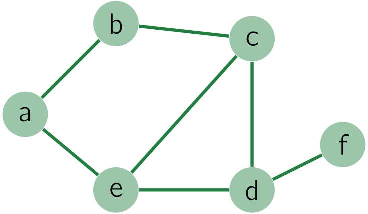

Independent sets include $\emptyset$, $\{a, c\}$, and $\{b, e, f\}$.

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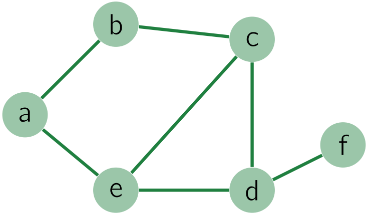

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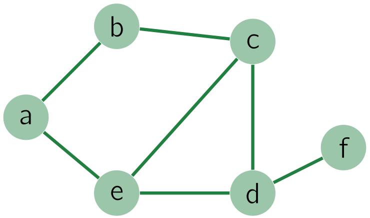

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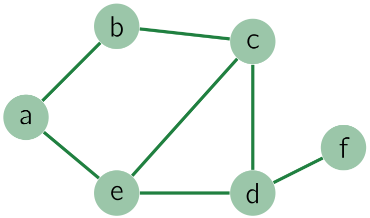

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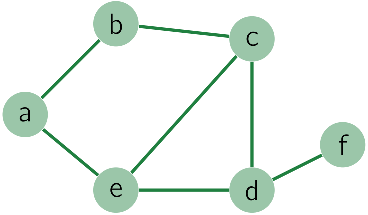

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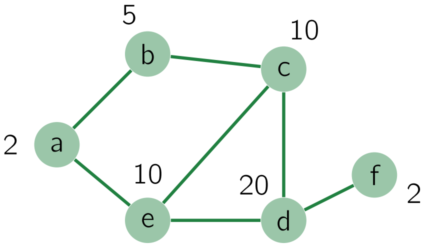

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Proof.

For any set S independent in G, either $v \notin S$ or $v \in S$.

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Any set S independent in G must be of the above two cases. Now maximize. □

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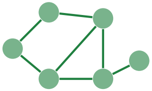

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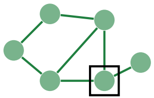

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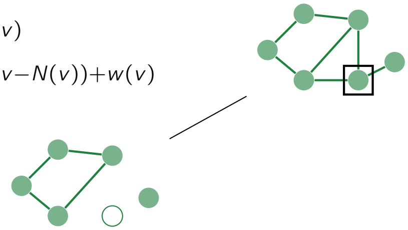

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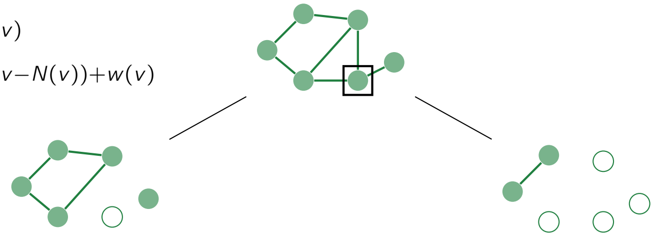

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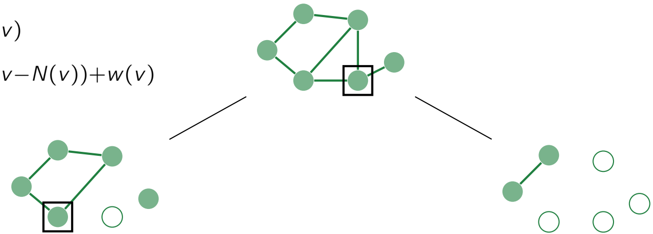

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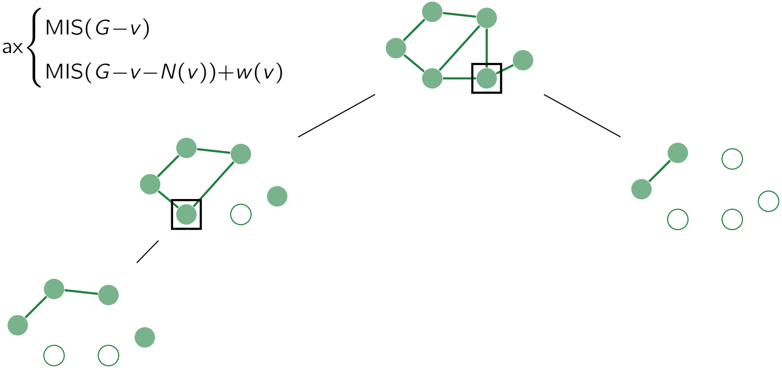

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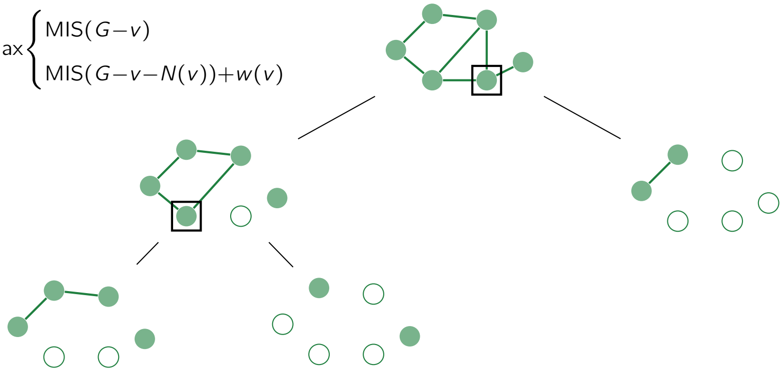

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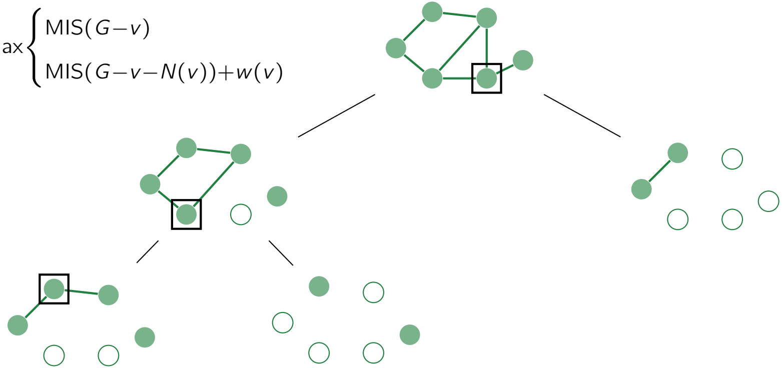

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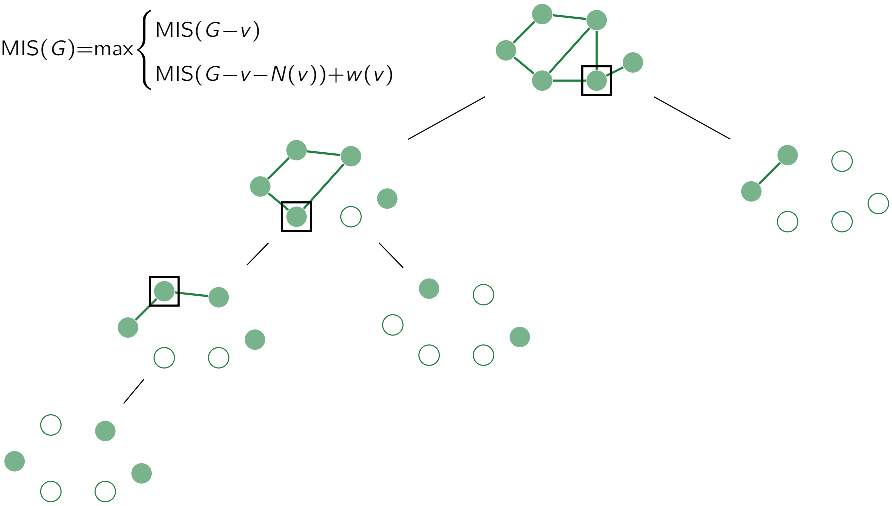

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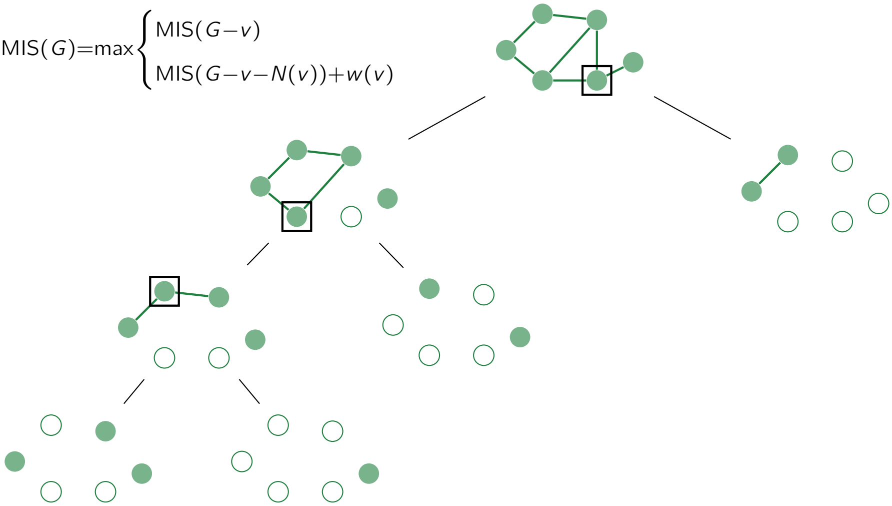

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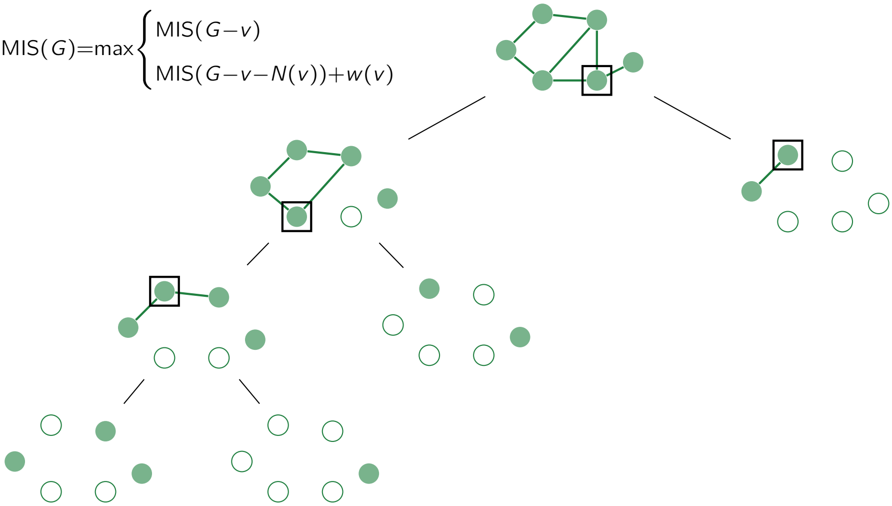

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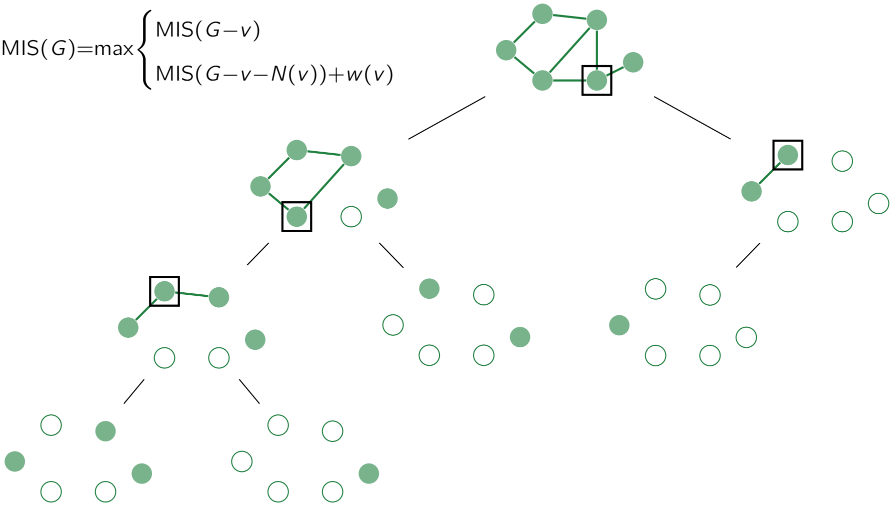

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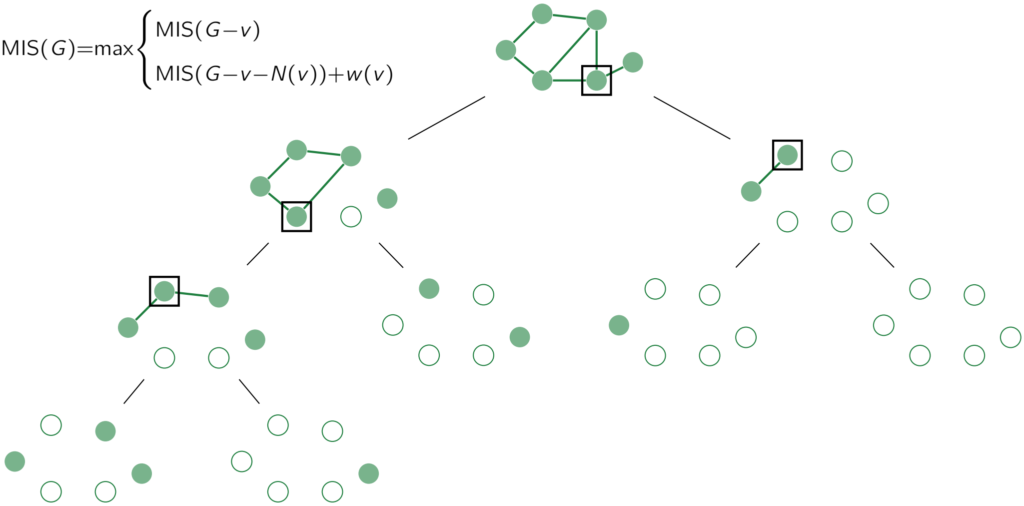

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$\texttt{recursive-MIS}(G = (V, E), w : V \to \mathbb{N})\texttt{:}$

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- memoization does not obviously help — subproblems correspond to subgraphs, of which there are possibly exponentially many

Maximum Independent Set, in Trees

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question:

Maximum Independent Set, in Trees

question: maximum weight independent set,

Maximum Independent Set, in Trees

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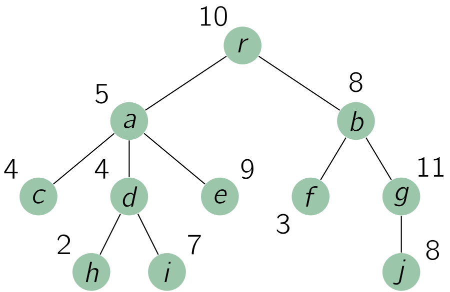

Maximum Independent Set, in Trees

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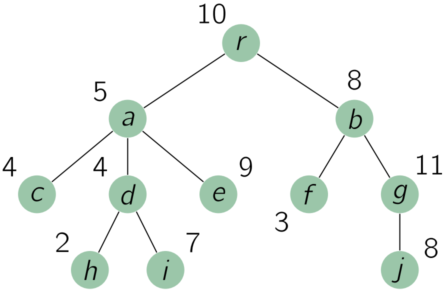

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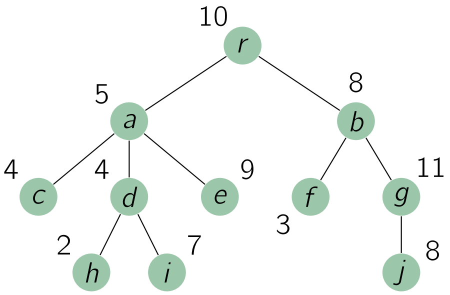

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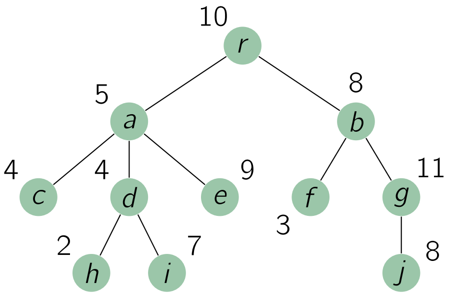

question:

- how to bound the number of subproblems in recursive algorithm?
- how to pick which vertex $v \in V$ to eliminate?

Maximum Independent Set, in Trees (II)

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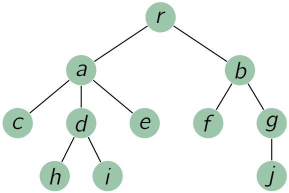

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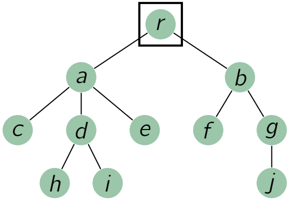

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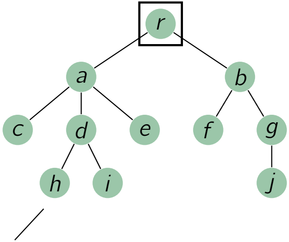

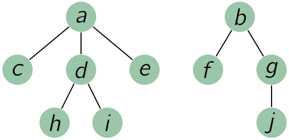

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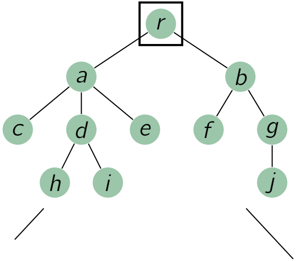

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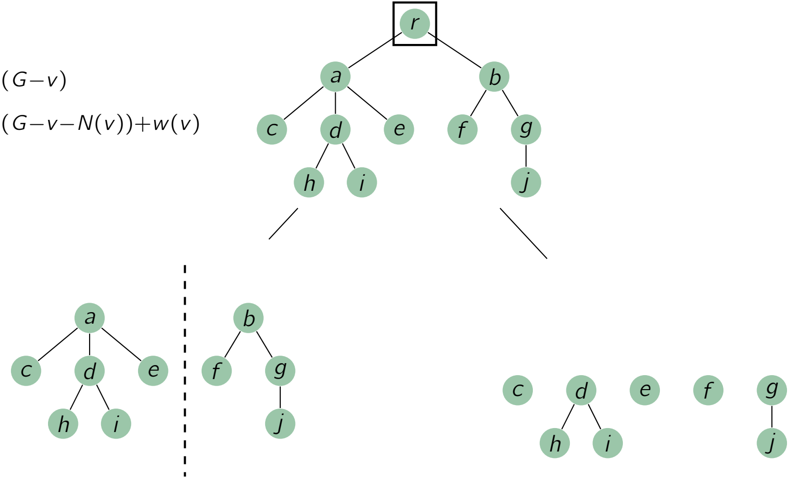

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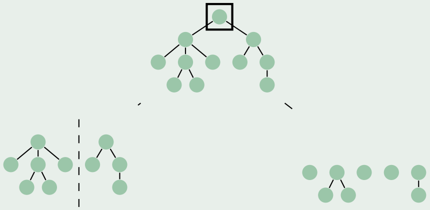

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$\Longrightarrow$ *memoized recursive algorithm is efficient*

Maximum Independent Set, in Trees (IV)

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dependency graph:

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$\Longrightarrow$ iterating over V in post-order traversal of T will satisfy the dependency graph

Maximum Independent Set, in Trees (V)

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iterative algorithm:

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complexity:

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$\texttt{iter-MIS-tree}(T = (V, E), w : V \to \mathbb{N})\texttt{:}$

let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of T

$\implies$ v_n is the root

for $1 \leq i \leq n$

$$M[i] = \max \begin{cases} \sum_{j : v_j \in N(v_i)} M[j] \\ \left(\sum_{j : v_j \in N(N(v_i))} M[j] \right) + w(v_i) \end{cases}$$

return $M[n]$

correctness: clear

complexity:

- $O(n)$ space to store $M[\cdot]$
- time
  - *naive:* $O(n)$ time per node, n nodes $\implies$ $O(n^2)$

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question: why does dynamic programming work on trees?

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- every tree T has a balanced separator consisting of a single node
- dynamic-programming + small balanced separators $\implies 2^{O(\sqrt{n})}$-time MIS algorithm for *planar* graphs

Minimum Dominating Set

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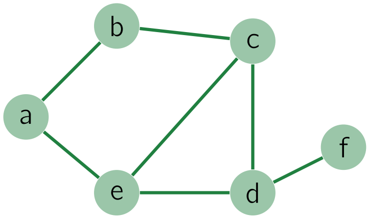

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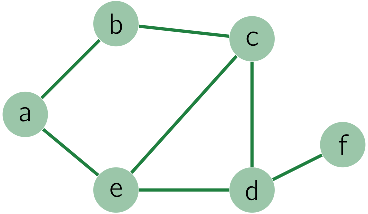

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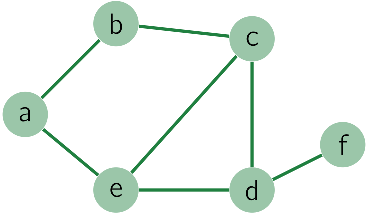

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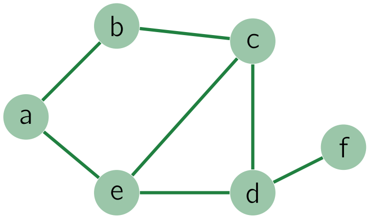

Dominating sets include $\{a, b, c, d, e, f\}$, $\{e, c, f\}$, and $\{a, b, f\}$.

Minimum Dominating Set (II)

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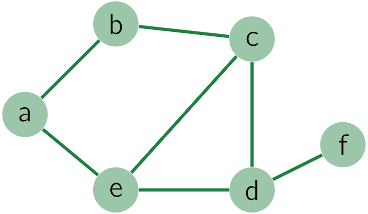

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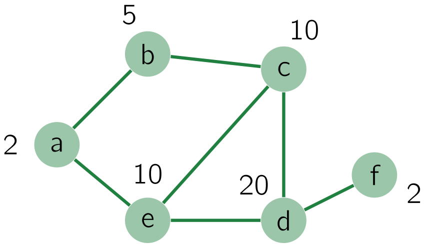

Minimum Dominating Set (III)

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remarks:

- minimum (weight) dominating set is solvable via brute force: try *all* possible subsets $\implies$ solvable in time $O(n^{O(1)}2^n)$
- no efficient algorithm *currently* known
- minimum weight dominating set is NP-hard $\implies$ an efficient algorithm *not* expected to exist
- minimum weight dominating set is efficiently solvable if the underlying graph is a *tree*

Minimum Dominating Set, in Trees

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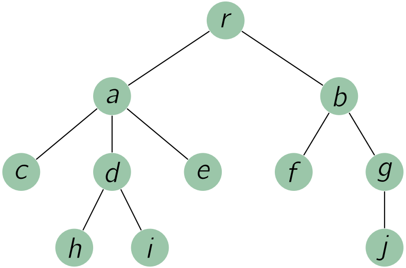

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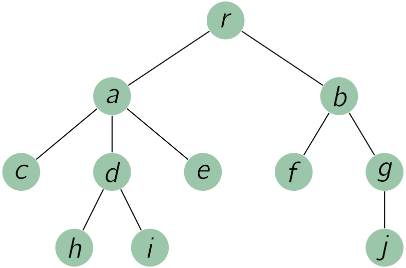

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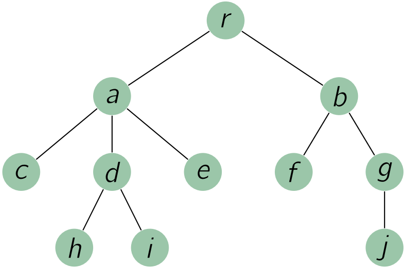

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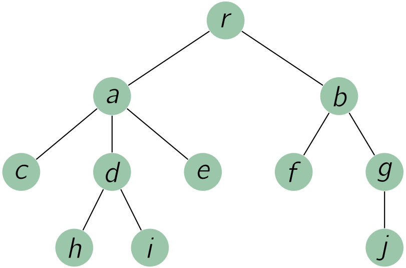

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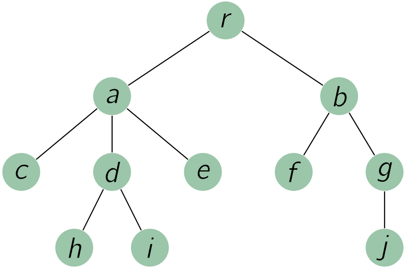

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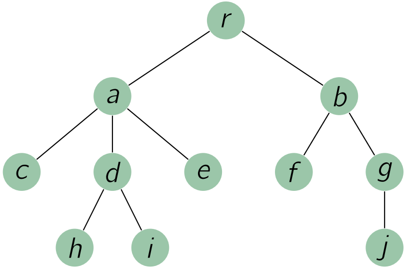

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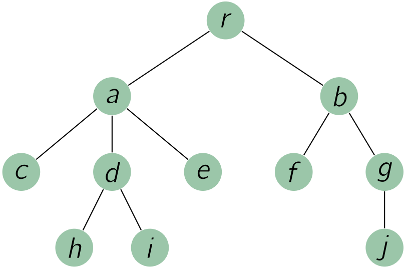

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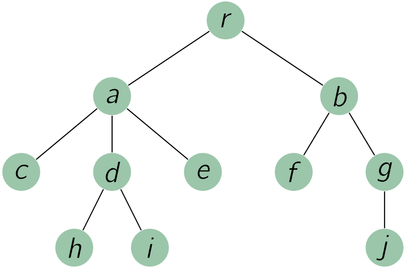

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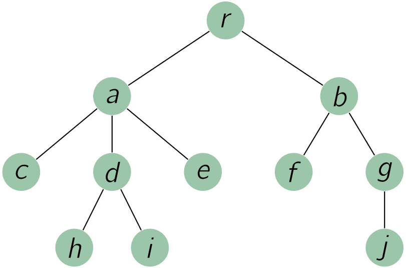

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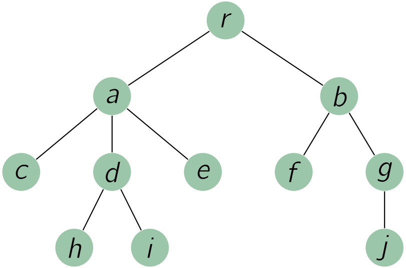

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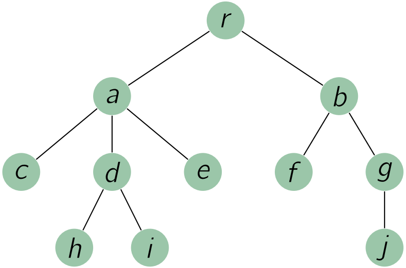

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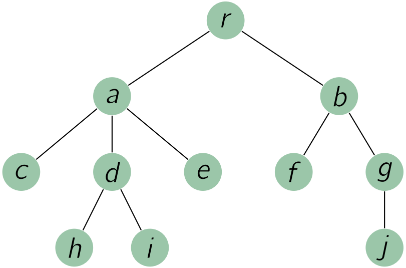
Minimum Dominating Set, in Trees

question: copy&paste from MIS on trees?



Let $T(v)$ denote the subtree rooted at $v \in V$, and let $S(v)$ be any minimum weight dominating set for $T(v)$.

building $S(r)$:

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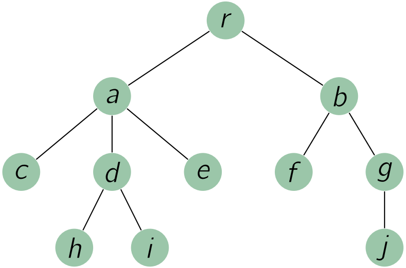

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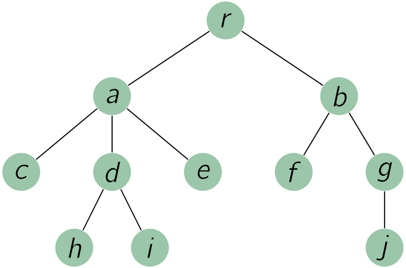

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question: how to parameterize these subproblems?

Minimum Dominating Set, in Trees (II)

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  - for $S = \emptyset$, $\min_{v \in S} f(v) = \infty$

Minimum Dominating Set, in Trees (III)

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T rooted tree with root r.

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T rooted tree with root r. $T(v)$ is subtree rooted at v.

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Minimum Dominating Set, in Trees (III)

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- **type-0**: regular dominating set
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Minimum Dominating Set, in Trees (III)

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Lemma

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$$\mathrm{OPT}_0(r) = \min$$

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Lemma

$$\mathrm{OPT}_0(r) = \min \begin{cases} \left(\sum_{v \in N(r)} \mathrm{OPT}_2(v) \right) \\ \end{cases}$$

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$$\mathrm{OPT}_0(r) = \min \begin{cases} \left(\sum_{v \in N(r)} \mathrm{OPT}_2(v)\right) + w(r) \\ \end{cases}$$

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Proof.

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Proof.

- in optimum S, $r \in S$

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Proof.

- in optimum S, $r \in S$
- in optimum S, $r \notin S$

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Proof.

- in optimum S, $r \in S$
- in optimum S, $r \notin S$ and r dominated by child $v \in S$ □

Minimum Dominating Set, in Trees (IV)

Minimum Dominating Set, in Trees (IV)

T rooted tree with root r. $T(v)$ is subtree rooted at v.

- **type-0**: regular dominating set
- **type-1**: dominating set which includes root r
- **type-2**: dominating set which is relaxed at root r

Minimum Dominating Set, in Trees (IV)

T rooted tree with root r. $T(v)$ is subtree rooted at v.

- **type-0**: regular dominating set
- **type-1**: dominating set which includes root r
- **type-2**: dominating set which is relaxed at root r

Lemma

Minimum Dominating Set, in Trees (IV)

T rooted tree with root r. $T(v)$ is subtree rooted at v.

- **type-0**: regular dominating set
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Lemma

$$\mathrm{OPT}_1(r) =$$

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- **type-0**: regular dominating set
- **type-1**: dominating set which includes root r
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Lemma

$$\mathrm{OPT}_1(r) = \left(\sum_{v \in N(r)} \right.$$

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T rooted tree with root r. $T(v)$ is subtree rooted at v.

- **type-0**: regular dominating set
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T rooted tree with root r. $T(v)$ is subtree rooted at v.

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Lemma

$$\mathrm{OPT}_1(r) = \left(\sum_{v \in N(r)} \mathrm{OPT}_2(v) \right) + w(r) \,.$$

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$$\mathrm{OPT}_1(r) = \left(\sum_{v \in N(r)} \mathrm{OPT}_2(v) \right) + w(r) \ .$$

Proof.

In optimum S, $r \in S$. □

Minimum Dominating Set, in Trees (V)

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T rooted tree with root r. $T(v)$ is subtree rooted at v.

- **type-0**: regular dominating set
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Minimum Dominating Set, in Trees (V)

T rooted tree with root r. $T(v)$ is subtree rooted at v.

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Minimum Dominating Set, in Trees (V)

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Lemma

$$\mathrm{OPT}_2(r) = \min$$

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Lemma

$$\mathrm{OPT}_2(r) = \min \begin{cases} \left(\sum_{v \in N(r)} \mathrm{OPT}_2(v)\right) + w(r) \\ \sum_{v \in N(r)} \end{cases}$$

Minimum Dominating Set, in Trees (V)

T rooted tree with root r. $T(v)$ is subtree rooted at v.

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- in optimum S, $r \notin S$ and r does not need to be dominated by children □

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details are an **exercise**

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- trees can be easily decomposed into a (small) number of subtrees, this allows a small number of resulting subproblems
- dynamic programming on trees can often be generalized to graphs of small *treewidth*

Overview (II)

today:

- dynamic programming *on trees*
- maximum independent set
- dominating set

next lecture:

- *more* dynamic programming

logistics:

- pset1 out, due R5 — can submit in *groups* of ≤ 3

TOC