cs473: Algorithms

Lecture 3: Dynamic Programming

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University of Illinois at Urbana-Champaign

September 2, 2019

Today

Today

logistics:

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- pset0 due R5,

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- pset0 due R5, (aka, tomorrow)

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example (Karatsuba, Strassen, ...):

- reduce problem instances of size n to problem instances of size $n/2$
- terminate recursion at $O(1)$-size problem instances, solve straightforwardly as a *base case*

Recursion (II)

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recursive paradigms:

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- **tail recursion**:

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- **dynamic programming:** expend effort to reduce given problem to multiple *correlated* smaller problems. Naive recursion often *not* efficient, use **memoization** to avoid wasteful recomputation.

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$\texttt{foo}(X)$

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    if X is a base case
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- *recursion tree:*

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analysis:

- *recursion tree:* each instance X spawns *new* children X_1, X_2, X_3
- *dependency graph:* each instance X links to sub-problems X_1, X_2, X_3

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Definition (Fibonacci 1200,)

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- $\implies 1 - \varphi \approx -.618\cdots \implies |(1-\varphi)^n| \leq 1$, and further $(1-\varphi)^n \to_{n\to\infty} 0$ $\implies F_n = \Theta(\varphi^n)$.

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complexity: let $T(n)$ denote the number of *additions*.

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Fibonacci Numbers (III)

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recursion tree:

Fibonacci Numbers (III)

recursion tree: for F_4

Fibonacci Numbers (III)

recursion tree: for F_4

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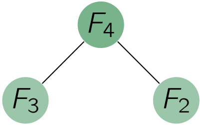

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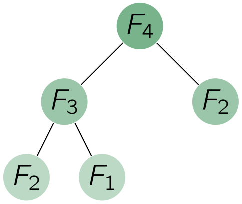

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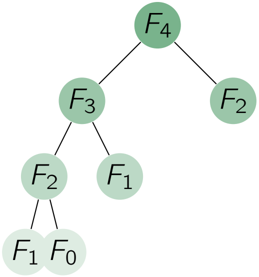

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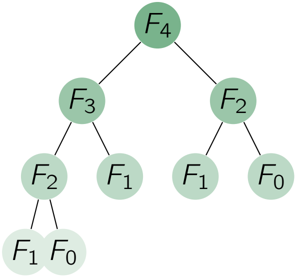

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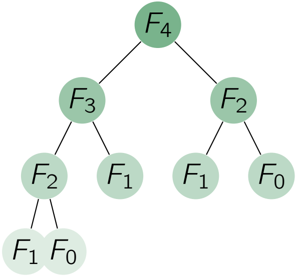

dependency graph:

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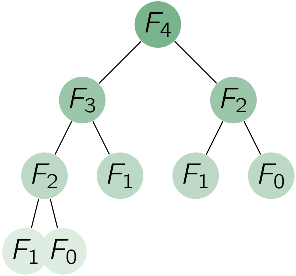

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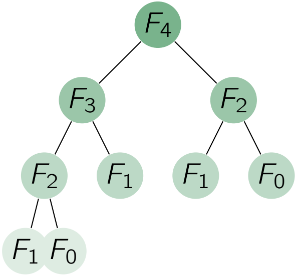

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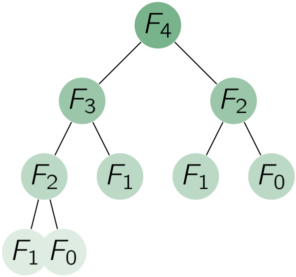

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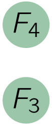

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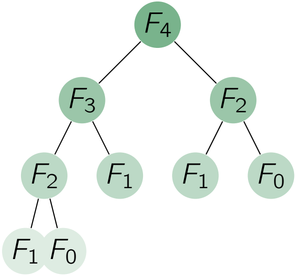

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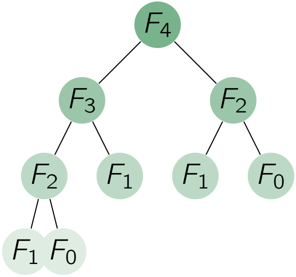

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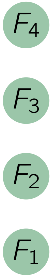

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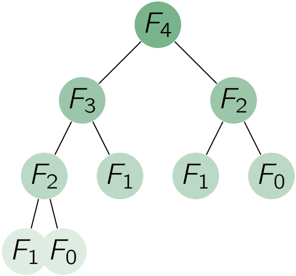

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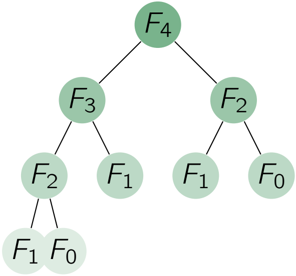

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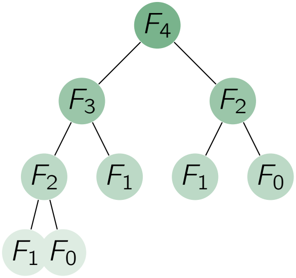

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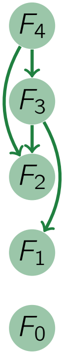

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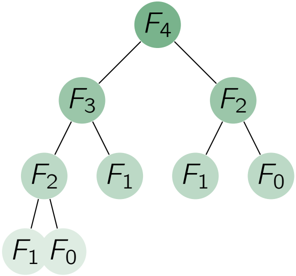

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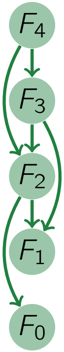

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- $F_n = \Theta(\varphi^n) \implies F_n$ takes $\Theta(n)$ bits $\implies$ each addition takes $\Theta(n)$ steps

$\implies$ $O(n^2)$ is the *actual* runtime

Memoization

recursive paradigms for F_n:

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- **naive recursion:**

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Dynamic programming is the method of speeding up naive recursion through memoization.

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- Memoizing a recursive algorithm is done by tracing through the dependency graph

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- *implicitly:* allow clever data structures to do this automatically

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global $\texttt{F}[\cdot]$

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  - requires *less* deliberation on problem structure, and can be less efficient
  - sometimes can be done automatically by functional programming languages (LISP, etc.)

Fibonacci Numbers (V)

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question: how much *space* do we need to memoize?

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fib-iter(n):
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correctness: clear

complexity: $O(n)$ additions, $O(1)$ numbers stored $\implies$ $O(n)$ *bits* stored

Memoization (IV)

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Definition

Dynamic programming is the method of speeding up naive recursion through memoization.

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Dynamic programming is the method of speeding up naive recursion through memoization.

goals:

- Given a recursive algorithm, analyze the complexity of its memoized version.
- Find the *right* recursion that can be memoized.
- Recognize when dynamic programming will efficiently solve a problem.
- Further optimize time- and space-complexity of dynamic programming algorithms.

Edit Distance

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Let $x, y \in \Sigma^\star$ be two strings over the alphabet Σ.

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m̲oney $\rightarrow$ boney̲ $\rightarrow$ bone̲ $\rightarrow$ bon̲a $\rightarrow$

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Example

money $\to$ boney $\to$ bone $\to$ bona $\to$ bo_a $\to$

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remarks:

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- edit distance ≤ 4

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remarks:

- edit distance ≤ 4
- intermediate strings can be arbitrary in $\Sigma^\star$

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The **cost** of an alignment is the number of pairs (i, j) where $x_i \neq y_j$.

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- can *also* ask to compute the alignment itself

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- can *also* ask to compute the alignment itself
- widely solved in practice,

Edit Distance (III)

question: given two strings $x, y \in \Sigma^\star$, compute their edit distance

Lemma

The edit distance between two strings $x, y \in \Sigma^\star$ is the minimum cost of an alignment.

Proof.

Exercise. □

question: given two strings $x, y \in \Sigma^\star$, compute the minimum cost of an alignment

remarks:

- can *also* ask to compute the alignment itself
- widely solved in practice, e.g., the BLAST heuristic for DNA edit distance

Edit Distance (IV)

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Lemma

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Let $x, y \in \Sigma^*$ be strings,

Edit Distance (IV)

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Let $x, y \in \Sigma^*$ *be strings, and* $a, b \in \Sigma$ *be symbols.*

Edit Distance (IV)

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Edit Distance (IV)

Lemma

Let $x, y \in \Sigma^*$ *be strings, and* $a, b \in \Sigma$ *be symbols. Then*

$$\mathrm{dist}(x \circ a, y \circ b) = \min \begin{cases} \mathrm{dist}(x, y) + \mathbb{1}[\![a \neq b]\!] \\ \\ \\ \end{cases}$$

Edit Distance (IV)

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In an optimal alignment from $x \circ a$ to $y \circ b$, either:

- a aligns to b, with cost $\mathbb{1}[\![a \neq b]\!]$
- a is deleted,

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Edit Distance (V)

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recursive algorithm:

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`dist(`x

Edit Distance (V)

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$$\texttt{dist}(x = x_1 x_2 \cdots x_n,$$

Edit Distance (V)

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Edit Distance (V)

recursive algorithm:

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Edit Distance (V)

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Edit Distance (V)

recursive algorithm:

$\texttt{dist}(x = x_1 x_2 \cdots x_n, y = y_1 y_2 \cdots y_m)$

if $n = 0$, **return** m

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Edit Distance (V)

recursive algorithm:

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\begin{aligned}
&\texttt{dist}(x = x_1 x_2 \cdots x_n, y = y_1 y_2 \cdots y_m) \\
&\quad \textbf{if } \ n = 0\,, \ \textbf{return } \ m \\
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&\quad d_1 = \textbf{dist}(x_{<n}, y_{<m}) + \mathbb{1}[\![x_n \neq y_m]\!]
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Edit Distance (V)

recursive algorithm:

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correctness:

Edit Distance (V)

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correctness: clear

Edit Distance (V)

recursive algorithm:

$$
\begin{array}{l}
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\end{array}
$$

correctness: clear

complexity:

Edit Distance (V)

recursive algorithm:

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\end{aligned}$$

correctness: clear

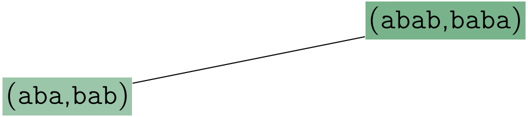
complexity: ???

Edit Distance (VI)

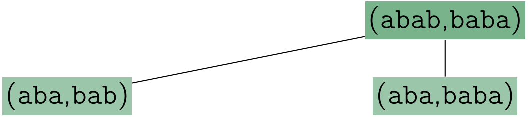
Edit Distance (VI)

(abab,baba)

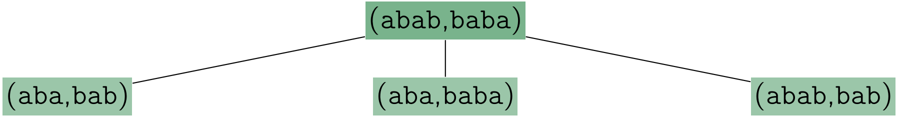
Edit Distance (VI)



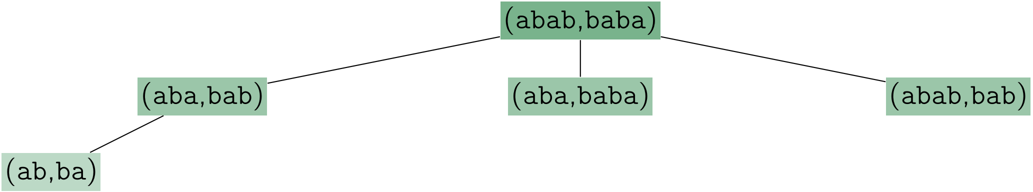
Edit Distance (VI)



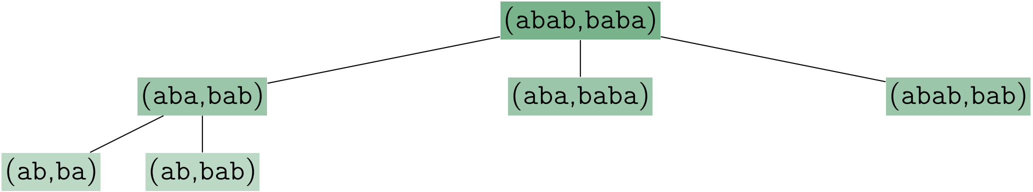
Edit Distance (VI)



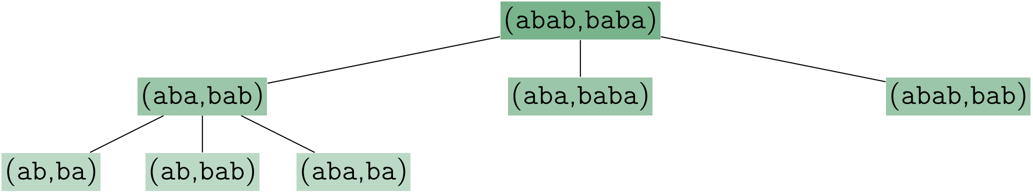
Edit Distance (VI)



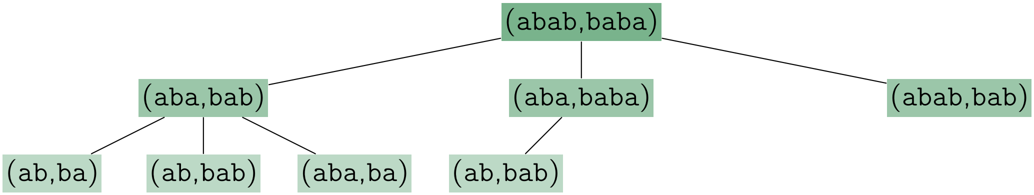
Edit Distance (VI)



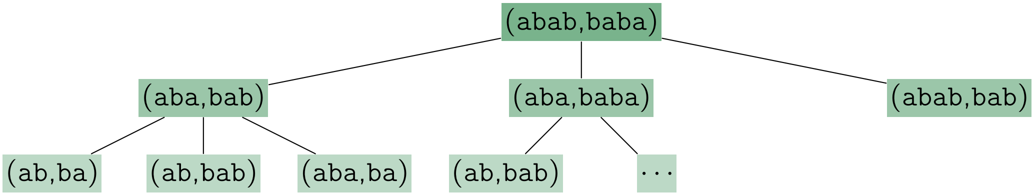
Edit Distance (VI)



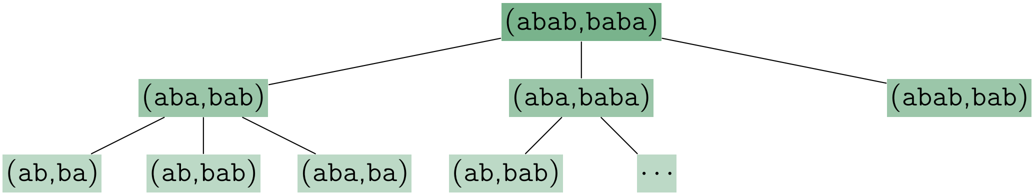
Edit Distance (VI)



Edit Distance (VI)

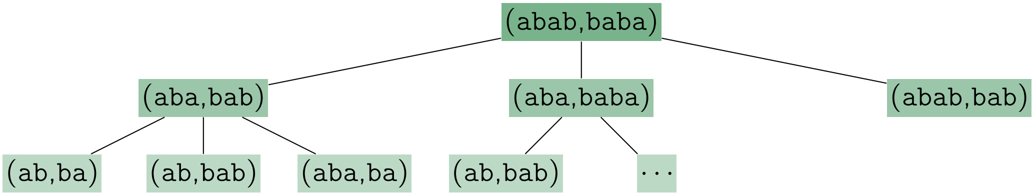


Edit Distance (VI)



(ab,bab) is *repeated*!

Edit Distance (VI)



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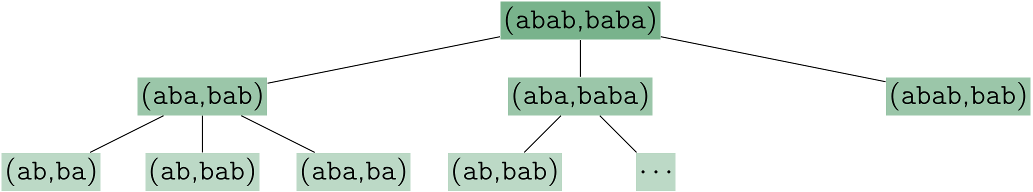
memoization:

Edit Distance (VI)



(ab,bab) is *repeated*!

memoization: define subproblem (i, j) as computing $\text{dist}(x_{\leq i}, y_{\leq j})$

Edit Distance (VII)

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memoized algorithm:

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global $d[\cdot][\cdot]$

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`dist(`$x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_m,$

Edit Distance (VII)

memoized algorithm:

global $d[\cdot][\cdot]$

`dist`$(x_1x_2\cdots x_n, y_1y_2\cdots y_m, (i,j))$

Edit Distance (VII)

memoized algorithm:

global $d[\cdot][\cdot]$
`dist`$(x_1x_2\cdots x_n, y_1y_2\cdots y_m, (i,j))$
  if $d[i][j]$ initialized

Edit Distance (VII)

memoized algorithm:

```
global d[·][·]
dist(x_1 x_2 ··· x_n, y_1 y_2 ··· y_m, (i, j))
    if d[i][j] initialized
        return d[i][j]
```

Edit Distance (VII)

memoized algorithm:

global $d[\cdot][\cdot]$

`dist`$(x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_m, (i, j))$

    if $d[i][j]$ initialized

        return $d[i][j]$

    if $i = 0$

Edit Distance (VII)

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$\texttt{dist}(x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_m, (i, j))$

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  if $d[i][j]$ initialized
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    $d[i][j] = j$
  elif $j = 0$

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>   **if** $i = 0$
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>   **elif** $j = 0$
> $d[i][j] = i$
>   **else**
> $d_1 = \textbf{dist}(x, y, (i-1, j-1)) + \mathbb{1}[\![x_i \neq y_j]\!]$

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Edit Distance (VII)

memoized algorithm:

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    if d[i][j] initialized
        return d[i][j]
    if i = 0
        d[i][j] = j
    elif j = 0
        d[i][j] = i
    else
        d1 = dist(x, y, (i − 1, j − 1)) + 𝟙[[xi ≠ yj]]
        d2 = dist(x, y, (i − 1, j)) + 1
        d3 = dist(x, y, (i, j − 1)) + 1
        d[i][j] = min(d1, d2, d3)
    return d[i][j]
```

memoized algorithm:

global $d[\cdot][\cdot]$

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$d[i][j] = \min(d_1, d_2, d_3)$

return $d[i][j]$

Edit Distance (VIII)

Edit Distance (VIII)

dependency graph:

Edit Distance (VIII)

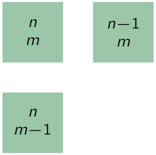
dependency graph:

Edit Distance (VIII)

dependency graph:

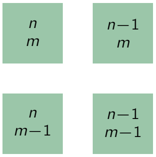
Edit Distance (VIII)

dependency graph:



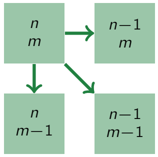
Edit Distance (VIII)

dependency graph:



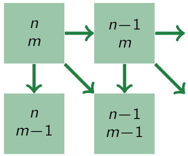
Edit Distance (VIII)

dependency graph:



Edit Distance (VIII)

dependency graph:



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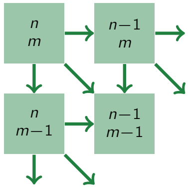

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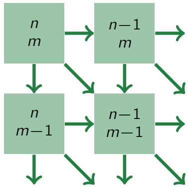

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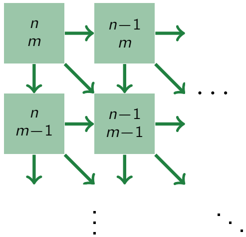

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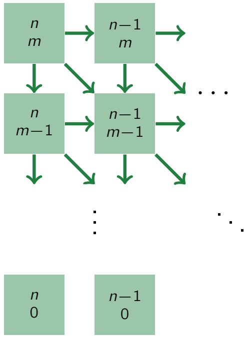
dependency graph:



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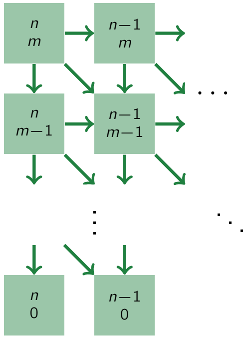

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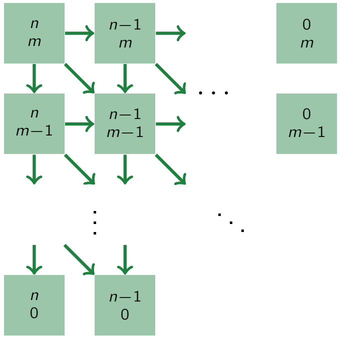

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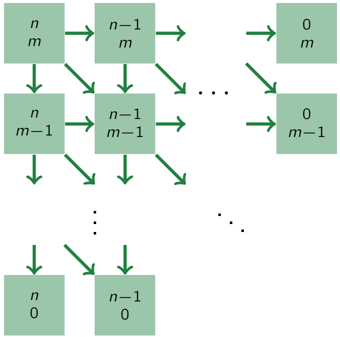

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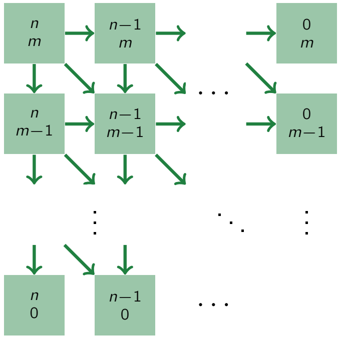

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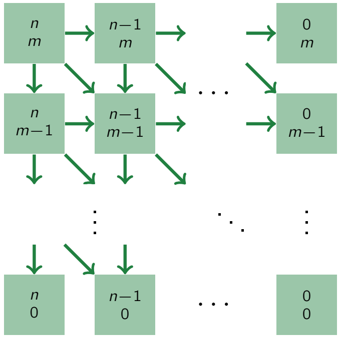

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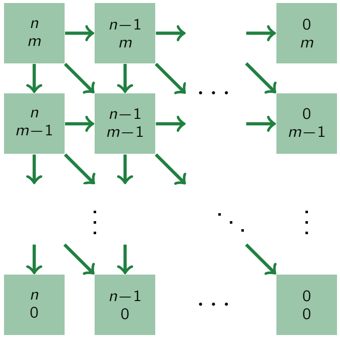

Edit Distance (IX)

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iterative algorithm:

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$\texttt{dist}(x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_m)$

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iterative algorithm:

$\texttt{dist}(x_1x_2\cdots x_n, y_1y_2\cdots y_m)$

  for $0 \le i \le n$

Edit Distance (IX)

iterative algorithm:

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dist(x_1 x_2 ··· x_n, y_1 y_2 ··· y_m)
    for 0 ≤ i ≤ n
        d[i][0] = i
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iterative algorithm:

$\texttt{dist}(x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_m)$

for $0 \leq i \leq n$

$d[i][0] = i$

for $0 \leq j \leq m$

$d[0][j] = j$

for $0 \leq i \leq n$

for $0 \leq j \leq m$

$$d[i][j] = \min \left\{ \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$$

Edit Distance (IX)

iterative algorithm:

$\texttt{dist}(x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_m)$

for $0 \le i \le n$

$d[i][0] = i$

for $0 \le j \le m$

$d[0][j] = j$

for $0 \le i \le n$

for $0 \le j \le m$

$$d[i][j] = \min \begin{cases} d[i-1][j-1] + \mathbb{1}[\![x_i \neq y_j]\!] \\ \\ \end{cases}$$

Edit Distance (IX)

iterative algorithm:

$\texttt{dist}(x_1x_2\cdots x_n, y_1y_2\cdots y_m)$

for $0 \le i \le n$

$d[i][0] = i$

for $0 \le j \le m$

$d[0][j] = j$

for $0 \le i \le n$

for $0 \le j \le m$

$$d[i][j] = \min \begin{cases} d[i-1][j-1] + \mathbb{1}[\![x_i \neq y_j]\!] \\ d[i-1][j] + 1 \\ \end{cases}$$

Edit Distance (IX)

iterative algorithm:

$\texttt{dist}(x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_m)$

for $0 \leq i \leq n$

$d[i][0] = i$

for $0 \leq j \leq m$

$d[0][j] = j$

for $0 \leq i \leq n$

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Edit Distance (IX)

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```
dist(x_1 x_2 ··· x_n, y_1 y_2 ··· y_m)
```

$\texttt{dist}(x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_m)$

for $0 \le i \le n$

$\quad d[i][0] = i$

for $0 \le j \le m$

$\quad d[0][j] = j$

for $0 \le i \le n$

$\quad$ **for** $0 \le j \le m$

$$d[i][j] = \min \begin{cases} d[i-1][j-1] + \mathbb{1}[\![x_i \neq y_j]\!] \\ d[i-1][j] + 1 \\ d[i][j-1] + 1 \end{cases}$$

return $d[n][m]$

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    for 0 ≤ i ≤ n
        d[i][0] = i
    for 0 ≤ j ≤ m
        d[0][j] = j
    for 0 ≤ i ≤ n
        for 0 ≤ j ≤ m
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$$d[i][j] = \min \begin{cases} d[i-1][j-1] + \mathbb{1}[\![x_i \neq y_j]\!] \\ d[i-1][j] + 1 \\ d[i][j-1] + 1 \end{cases}$$

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correctness:

Edit Distance (IX)

iterative algorithm:

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        d[i][0] = i
    for 0 ≤ j ≤ m
        d[0][j] = j
    for 0 ≤ i ≤ n
        for 0 ≤ j ≤ m
            d[i][j] = min { d[i−1][j−1] + 𝟙⟦x_i ≠ y_j⟧
                            d[i−1][j] + 1
                            d[i][j−1] + 1
    return d[n][m]
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$$d[i][j] = \min \begin{cases} d[i-1][j-1] + \mathbb{1}[\![x_i \neq y_j]\!] \\ d[i-1][j] + 1 \\ d[i][j-1] + 1 \end{cases}$$

correctness: clear

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    for 0 ≤ i ≤ n
        for 0 ≤ j ≤ m
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```
    return d[n][m]
```

correctness: clear

complexity:

Edit Distance (IX)

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    for 0 ≤ i ≤ n
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$$d[i][j] = \min \begin{cases} d[i-1][j-1] + \mathbb{1}[\![x_i \neq y_j]\!] \\ d[i-1][j] + 1 \\ d[i][j-1] + 1 \end{cases}$$

correctness: clear

complexity: $O(nm)$ time,

Edit Distance (IX)

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    for 0 ≤ i ≤ n
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correctness: clear

complexity: $O(nm)$ time, $O(nm)$ space

Edit Distance (X)

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Corollary

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Given two strings $x, y \in \Sigma^\star$ can compute the minimum cost alignment

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Given two strings $x, y \in \Sigma^\star$ can compute the minimum cost alignment in $O(nm)$-time and $O(nm)$-space.

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Given two strings $x, y \in \Sigma^\star$ *can compute the minimum cost alignment in* $O(nm)$*-time and* $O(nm)$*-space.*

Proof.

Edit Distance (X)

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Exercise.

Edit Distance (X)

Corollary

Given two strings $x, y \in \Sigma^\star$ can compute the minimum cost alignment in $O(nm)$-time and $O(nm)$-space.

Proof.

Exercise. *Hint:*

Edit Distance (X)

Corollary

Given two strings $x, y \in \Sigma^\star$ *can compute the minimum cost alignment in* $O(nm)$*-time and* $O(nm)$*-space.*

Proof.

Exercise. *Hint:* follow *how* each subproblem was solved.

Edit Distance (X)

Corollary

Given two strings $x, y \in \Sigma^\star$ can compute the minimum cost alignment in $O(nm)$-time and $O(nm)$-space.

Proof.

Exercise. *Hint:* follow *how* each subproblem was solved. $\square$

Dynamic Programming

template:

Dynamic Programming

template:

- develop recursive algorithm

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template:

- develop recursive algorithm
- understand structure of subproblems

Dynamic Programming

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- develop recursive algorithm
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- memoize

Dynamic Programming

template:

- develop recursive algorithm
- understand structure of subproblems
- memoize
  - implicitly,

Dynamic Programming

template:

- develop recursive algorithm
- understand structure of subproblems
- memoize
  - implicitly, via data structure

Dynamic Programming

template:

- develop recursive algorithm
- understand structure of subproblems
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Dynamic Programming

template:

- develop recursive algorithm
- understand structure of subproblems
- memoize
  - implicitly, via data structure
  - explicitly, converting to iterative algorithm

Dynamic Programming

template:

- develop recursive algorithm
- understand structure of subproblems
- memoize
  - implicitly, via data structure
  - explicitly, converting to iterative algorithm to traverse dependency graph

Dynamic Programming

template:

- develop recursive algorithm
- understand structure of subproblems
- memoize
  - implicitly, via data structure
  - explicitly, converting to iterative algorithm to traverse dependency graph via topological sort

Dynamic Programming

template:

- develop recursive algorithm
- understand structure of subproblems
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- analysis

Dynamic Programming

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- develop recursive algorithm
- understand structure of subproblems
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- analysis (time,

Dynamic Programming

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- develop recursive algorithm
- understand structure of subproblems
- memoize
  - implicitly, via data structure
  - explicitly, converting to iterative algorithm to traverse dependency graph via topological sort
- analysis (time, space)

Dynamic Programming

template:

- develop recursive algorithm
- understand structure of subproblems
- memoize
  - implicitly, via data structure
  - explicitly, converting to iterative algorithm to traverse dependency graph via topological sort
- analysis (time, space)
- further optimization

Knapsack

Knapsack

the knapsack problem:

Knapsack

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input: knapsack capacity $W \in \mathbb{N}$

Knapsack

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input: knapsack capacity $W \in \mathbb{N}$ (in pounds).

Knapsack

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input: knapsack capacity $W \in \mathbb{N}$ (in pounds). n items with weights $w_1, \ldots, w_n \in \mathbb{N}$,

Knapsack

the knapsack problem:

input: knapsack capacity $W \in \mathbb{N}$ (in pounds). n items with weights $w_1, \ldots, w_n \in \mathbb{N}$, and values $v_1, \ldots, v_n \in \mathbb{N}$.

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input: knapsack capacity $W \in \mathbb{N}$ (in pounds). n items with weights $w_1, \ldots, w_n \in \mathbb{N}$, and values $v_1, \ldots, v_n \in \mathbb{N}$.

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$$\max_{S \subseteq [n]} \quad \sum_{i \in S} v_i$$

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$$\max_{\substack{S \subseteq [n] \\ \sum_{i \in S} w_i \leq W}} \sum_{i \in S} v_i$$

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remarks:

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remarks:

- prototypical problem in *combinatorial optimization*,

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- prototypical problem in *combinatorial optimization*, can be generalized in numerous ways

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remarks:

- prototypical problem in *combinatorial optimization*, can be generalized in numerous ways
- needs to be solved in practice

Knapsack (II)

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Example

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item	1	2	3	4	5
weight	1	2	5	6	7
value	1	6	18	22	28

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item	1	2	3	4	5
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For $W = 11$,

Knapsack (II)

Example

item	1	2	3	4	5
weight	1	2	5	6	7
value	1	6	18	22	28

For $W = 11$, the best is $\{3, 4\}$ giving value 40.

Knapsack (II)

Example

item	1	2	3	4	5
weight	1	2	5	6	7
value	1	6	18	22	28

For $W = 11$, the best is $\{3, 4\}$ giving value 40.

Definition

Knapsack (II)

Example

item	1	2	3	4	5
weight	1	2	5	6	7
value	1	6	18	22	28

For $W = 11$, the best is $\{3, 4\}$ giving value 40.

Definition

In the special case of when $v_i = w_i$ for all i,

Knapsack (II)

Example

item	1	2	3	4	5
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value	1	6	18	22	28

For $W = 11$, the best is $\{3, 4\}$ giving value 40.

Definition

In the special case of when $v_i = w_i$ for all i, the knapsack problem is called the **subset sum** problem.

Knapsack (III)

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item	1	2	3	4	5
value	1	6	16	22	28
weight	1	2	5	6	7

Knapsack (III)

item	1	2	3	4	5
value	1	6	16	22	28
weight	1	2	5	6	7

and weight limit $W = 15$.

Knapsack (III)

item	1	2	3	4	5
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and weight limit $W = 15$. What is the best solution value?

Knapsack (III)

item	1	2	3	4	5
value	1	6	16	22	28
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and weight limit $W = 15$. What is the best solution value?

(a) 22

(b) 28

(c) 38

(d) 50

(e) 56

Knapsack (IV)

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greedy approaches:

Knapsack (IV)

greedy approaches:

- greedily select by maximum value:

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- greedily select by maximum value:

item	1	2	3
value	2	2	3
weight	1	1	2

Knapsack (IV)

greedy approaches:

- greedily select by maximum value:

item	1	2	3
value	2	2	3
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For $W = 2$,

Knapsack (IV)

greedy approaches:

- greedily select by maximum value:

item	1	2	3
value	2	2	3
weight	1	1	2

For $W = 2$, greedy-value will pick $\{3\}$,

Knapsack (IV)

greedy approaches:

- greedily select by maximum value:

item	1	2	3
value	2	2	3
weight	1	1	2

  For $W = 2$, greedy-value will pick $\{3\}$, but optimal is $\{1, 2\}$.

Knapsack (IV)

greedy approaches:

- greedily select by maximum value:

item	1	2	3
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  For $W = 2$, greedy-value will pick $\{3\}$, but optimal is $\{1, 2\}$.

- greedily select by minimum weight:

Knapsack (IV)

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- greedily select by maximum value:

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weight	1	1	2

  For $W = 2$, greedy-value will pick $\{3\}$, but optimal is $\{1, 2\}$.

- greedily select by minimum weight:

item	1	2
value	1	3
weight	1	2

Knapsack (IV)

greedy approaches:

- greedily select by maximum value:

item	1	2	3
value	2	2	3
weight	1	1	2

  For $W = 2$, greedy-value will pick $\{3\}$, but optimal is $\{1, 2\}$.

- greedily select by minimum weight:

item	1	2
value	1	3
weight	1	2

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Knapsack (IV)

greedy approaches:

- greedily select by maximum value:

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value	3	3	5
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remark: while greedy algorithms fail to get the *best* result, they can still be useful for getting solutions that are *approximately* the best

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$\Longrightarrow$ from instance W, $v_1, \ldots, v_n$, and $w_1, \ldots, w_n$ we generate $O(n \cdot W)$-many subproblems $(i, w)_{i \in [n], w \leq W}$.

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      $M[i, w] = M[i-1, w]$

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Today

today:

- recursion
- dynamic programming
  - fibonacci numbers
  - edit distance
  - knapsack

next time: *more* dynamic programming **logistics:**

- pset0 due R5, (aka, tomorrow) — submit *individually*!
- pset1 out tomorrow, due R5 (next week)
- piazza signup

TOC