Michael Forbes
miforbes@illinois.edu

2019-09-17.4 → 2019-09-19.1
2019-09-19.2 ←
cs473

cs 473 Algorithms : Lecture 8

logistics
- pset 3 due W 10
- midterm 1 on Oct 7

last time
- probability review
- randomized algorithms
- quicksort

today
- quicksort
- concentration bounds

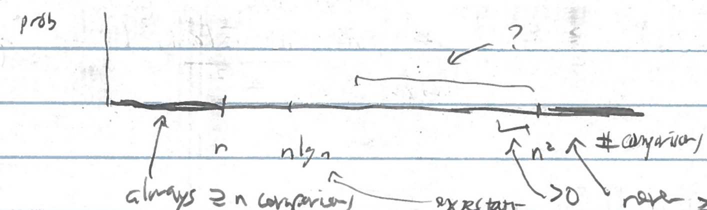
quicksort$(a_1, \ldots, a_n)$
  $i_0 = \text{rand}(n) + 1$
  pick pivot a_{i_0}
  partition $\bar{a} = (\bar{b}, a_{i_0}, \bar{c})$
    $b_j < a_{i_0} < c_k$
  return (quicksort$(\bar{b})$, a_{i_0}, quicksort$(\bar{c})$)

correctness: always $\Rightarrow$ las vegas

complexity: [[runtime]]
$T(\bar{a}) = \#$ comparisons made w/ input $\bar{a}$ ← random var

$$\overline{T}(n) = \max_{|\bar{a}|=n} \mathbb{E}[T(\bar{a})] \quad \text{[[worst case expected runtime]]}$$
$$\leq O(n) + \frac{1}{n}\sum_{i=1}^{n} (T(i-1) + T(n-i)) \quad \text{[[condition on rank}(a_{i_0}) \text{]]}$$
$$\leq O(n \lg n) \quad \text{[[pset 0]]}$$

Q: distribution of runtime?



Q: $\Pr[T(\bar{a}) \geq \frac{n^2}{10}]$
$\geq \frac{1}{2}$? no
$\leq \frac{1}{n}$ yes [[small]]
$\leq \frac{1}{2^{n^{\Omega(1)}}}$ yes [[really small]]

ex: algo: $X = \text{Geom}(1/2)$
  return $f(X)$ [[subroutine]]

$i \geq 1$ $\Pr[X = i] = \#$ coin flips needed to see heads $= 1/2^i$

$$\mathbb{E}[X] = \sum_\omega X(\omega) \cdot \Pr[\omega] = \sum_i i \cdot \frac{1}{2^i} = 2$$

$f(i)$ is subroutine takes 2^i steps

naive: expected runtime is $2^{\mathbb{E}[X]} = 2^2 = 4$

actual: $\mathbb{E}[Y] = \sum_{i \geq 1} \underbrace{\mathbb{E}[Y \mid X = i]}_{2^i} \cdot \underbrace{\Pr[X = i]}_{1/2^i} = \sum_i 1 = \infty$ [[exaggerated]]

rmk: expected runtime analysis not strong enough for analyzing subroutines

Q: (concentration bound) $\Pr[X \geq \mathbb{E}[X]] = ?$

lem (Markov's Inequality): $X \geq 0$ rand var. For $a \geq 0$ $\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$

Pf: $\mathbb{E}[X] = \underbrace{\mathbb{E}[X \mid X \geq a]}_{\geq a} \cdot \Pr[X \geq a] + \underbrace{\mathbb{E}[X \mid X < a]}_{\geq 0} \cdot \underbrace{\Pr[X < a]}_{\geq 0}$ ← $X \geq 0$

$\geq a \cdot \Pr[X \geq a]$ ∎

Michael Forbes
miforbes@illinois.edu
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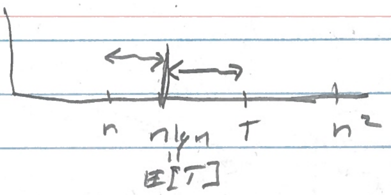
Cor: $X \ge 0$, $k \ge 0$. $\Pr[X \ge k \cdot \mathbb{E}[X]] \le \frac{1}{k}$

Cor: $\Pr[T(\vec{a}) \ge k \cdot \Theta(n \lg n)] \le \frac{1}{k}$ [$T(\vec{a}) \ge 0$]

$\ge n^2/10 \qquad \le O(\frac{\lg n}{n})$

rmk: - no hypothesis on X other than $X \ge 0$
[exer: find rv when this is an equality]
- is tight in general [to do better we need more structure]

def: X rand var. The variance is $Var(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$
[square so is positive]



lem: $\boxed{Var(X) =} \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2\mathbb{E}[X]\cdot X + \mathbb{E}[X]^2]$
$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\cdot\mathbb{E}[X] + \mathbb{E}[X]^2 \boxed{= \mathbb{E}[X^2] - \mathbb{E}[X]^2}$

prop (Chebyshev Inequality): $\varepsilon \ge 0$ $\Pr[|X - \mathbb{E}[X]| \ge \varepsilon] \le \frac{Var(X)}{\varepsilon^2}$

Pf: $= \Pr[\underbrace{(X-\mathbb{E}[X])^2}_{\text{nonneg}} \ge \varepsilon^2] \underset{\text{Markov}}{\le} \frac{\mathbb{E}[(X-\mathbb{E}[X])^2]}{\varepsilon^2} = \square$

lem: X, Y indep.
a) $\mathbb{E}[XY] = \mathbb{E}[X]\cdot\mathbb{E}[Y]$
b) $Var(X+Y) = Var(X) + Var(Y)$
c) $Var(a\cdot X) = a^2 \cdot Var(X)$

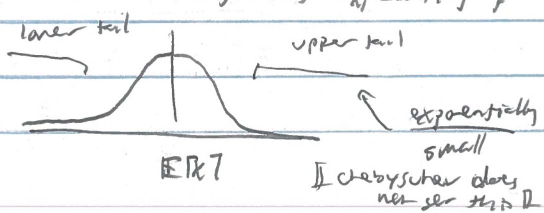
Pf: (c) $\mathbb{E}[(aX - \underbrace{\mathbb{E}[aX]}_{=a\cdot\mathbb{E}[X]})^2] = \mathbb{E}[(a(X-\mathbb{E}[X]))^2] = a^2 \cdot Var(X)$

b) $= \mathbb{E}[((X+Y) - \mathbb{E}[X+Y])^2] = \mathbb{E}[((X-\mathbb{E}[X]) + (Y-\mathbb{E}[Y]))^2]$
$= \underbrace{\mathbb{E}[(X-\mathbb{E}[X])^2]}_{=Var(X)} + 2\,\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] + \underbrace{\mathbb{E}[(Y-\mathbb{E}[Y])^2]}_{=Var(Y)}$
$\underset{[\text{indep}]}{=} \underbrace{\mathbb{E}[(X-\mathbb{E}[X])]}_{=0} \cdot \underbrace{\mathbb{E}[(Y-\mathbb{E}[Y])]}_{=0}$ $\square$

Prop: $X_1, \ldots, X_n \in [0,1]$ independent rand var, $\varepsilon \ge 0$, $X = \frac{X_1 + \cdots + X_n}{n}$. $\Pr[|X - \mathbb{E}[X]| \ge \varepsilon] \le \frac{1}{\varepsilon^2 n}$

Pf: $Var(X_i) = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 \le \mathbb{E}[X_i^2] \le 1$
$\Rightarrow Var(X) = \frac{1}{n^2} \cdot \sum \underbrace{Var(X_i)}_{\le 1} \le 1/n$. Chebyshev ⇒

Rmk: - fix ε. $\Pr[|X - \mathbb{E}[X]| \ge \varepsilon] \to 0$ as $n \to \infty$ ← (weak) law of large numbers [justifies "expectation"]
- say X_i identical. As $n \to \infty$
$X = \frac{X_1 + \cdots + X_n}{n} \to$ gaussian distribution



Michael Forbes
mforbes@illinois.edu

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thm (Chernoff Bound): $X_1, \ldots, X_n \in [0,1]$ indep, $X = \frac{X_1 + \cdots + X_n}{n}$, $\varepsilon \geq 0$

$$\Pr[\underbrace{X - \mathbb{E}[X] \geq \varepsilon}_{\text{upper tail}}] \leq e^{-KL(\mathbb{E}[X]+\varepsilon \| \mathbb{E}[X]) \cdot n}$$

↳ Kullback-Liebler divergence

Sketch: [more on pset 4]
$$= \Pr[r \cdot \sum X_i \geq r(\mathbb{E}[\sum X_i] + \varepsilon \cdot n)] \quad \text{for } r > 0$$
$$= \Pr[\underbrace{e^{r \cdot \sum X_i}}_{\text{non-neg}} \geq e^{r \cdot \sum \mathbb{E}[X_i] + r \cdot \varepsilon \cdot n}] \underset{\text{Markov}}{\leq} \frac{\mathbb{E}[\overbrace{e^{r \sum X_i}}^{=\prod e^{rX_i}}]}{e^{r \sum \mathbb{E}[X_i] + r\varepsilon n}} = \prod \mathbb{E}[e^{rX_i}]$$
↑ indep

Cor: $X = \frac{X_1 + \cdots + X_n}{n}$, $X_i \in [0,1]$ indep, $\varepsilon \geq 0$ $\leq e^{\cdots}$ ← optimize over r.

$$\Pr[|X - \mathbb{E}[X]| \geq \varepsilon] \leq 2 \cdot e^{-2\varepsilon^2 n} \quad \text{[additive]}$$

$$\Pr[X \geq (1+\varepsilon)\mathbb{E}[X]] \leq \begin{cases} e^{-\varepsilon^2 \cdot \mathbb{E}[X]/3} & \varepsilon \leq 1 \\ e^{-\varepsilon \mathbb{E}[X]/3} & \geq 1 \\ e^{-(1+\varepsilon)\ln(1+\varepsilon) \cdot \frac{\mathbb{E}[X]}{4}} & \geq 1 \end{cases} \quad \text{[multiplicative]}$$

$$\Pr[X \leq (1-\varepsilon)\mathbb{E}[X]] \leq e^{-\varepsilon^2 \mathbb{E}[X]/3} \quad \varepsilon \leq 1$$

[proven w/ Taylor series on KL, see pset 4]

ex: $f : \Sigma \to [0,1]$

want: $\frac{1}{|\Sigma|} \sum_\sigma f(\sigma) = \mu$ [true average opinion]

algo: pick $\sigma_1, \ldots, \sigma_n$ at random, return $\frac{f(\sigma_1) + \cdots + f(\sigma_n)}{n}$

complexity: $O(n)$ [what is n?]

correctness: $X_i = f(\sigma_i)$ $\quad \mathbb{E}[X_i] = \mu$

$X = \frac{\sum X_i}{n}$ $\quad \mathbb{E}[X] = \mu$

failure probability

$$\Pr[|X - \mu| \geq \underbrace{\varepsilon}_{\text{error}}] \underset{\text{[Chebyshev]}}{\leq} \frac{1}{\varepsilon^2 n} = \delta$$
$$\underset{\text{[Chernoff]}}{\leq} \underbrace{2e^{-\varepsilon^2 n}}_{} = \delta$$

↓ exponential improvement

eg: if $n = 738$, $\varepsilon = .05 \Rightarrow \leq .05$

ex: (balls into bins): m balls n bins, each ball thrown into random bin

$X_{ij} = \mathbb{1}[\text{ball } i \to \text{bin } j]$ $\quad Y_j := \sum X_{ij}$ ← # balls in bin j

$\mathbb{E}[X_{ij}] = 1/n$ $\quad \mathbb{E}[Y_j] = m/n$ [fundamental process in load balancing]

Prop: $m = n$. $Y = \max_j Y_j$. $\Pr[Y \geq k \cdot \frac{\ln n}{\ln \ln n}] \leq \frac{1}{n^{O(k)}}$

Pf: observe $X_{ij}, X_{i'j}$ independent $\in [0,1]$

$$\mathbb{E}[Y_j] = \sum_i \underbrace{\mathbb{E}[X_{ij}]}_{=1/n} = 1.$$

$$\Pr[Y_j \geq c \cdot 1] \underset{c \geq 2}{\leq} e^{-\frac{c \ln c}{4} \cdot 1} = e^{-\Theta(k \ln n)} = \frac{1}{n^{\Theta(k)}}$$

↑ [multiplicative Chernoff] $\quad c = k \cdot \frac{\ln n}{\ln \ln n}$

$$\Pr[\max_j Y_j \geq c] = \Pr[\exists_j\, Y_j \geq c] \underset{\text{[union]}}{\leq} \underbrace{\sum_j}_{n} \underbrace{\Pr[Y_j \geq c]}_{\leq \frac{1}{n^{\Theta(k)}}}$$

$$\leq n \cdot \frac{1}{n^{\Theta(k)}} = \frac{1}{n^{\Theta(k)}}.$$

Michael Forbes
mforbes@illinois.edu
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Cor $\mathbb{E}[\underbrace{\max_j Y_j}_{=Y}] \le \underbrace{\mathbb{E}[Y \mid Y \le k \cdot \frac{\ln n}{\ln\ln n}]}_{\le \frac{k \ln n}{\ln\ln n}} \cdot \underbrace{\Pr[\quad]}_{\le 1}$

$+ \underbrace{\Pr[Y \mid Y > k \frac{\ln n}{\ln\ln n}]}_{\le n} \cdot \underbrace{\Pr[\quad]}_{= \frac{1}{n^{\Theta(k)}}}$

w/ $k = \Theta(1)$ $\le O\left(\frac{\ln n}{\ln\ln n}\right)$

Rmk: - ~~Chernoff + Union is very powerful~~
- is tight

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quicksort($\bar{a}$)
  pivot a_{i_0}
  return (quicksort($\bar{b}$), a_{i_0}, quicksort($\bar{c}$))   [[saw Markov concentration]]

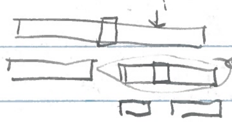
Prop: $\mathbb{E}[T(\bar{a})] \le O(n \lg n)$ [[concentration w/ Chernoff?]]

Pf: define $D(\bar{a})$ to be the maximum recursion depth
  then $T(\bar{a}) \le O(D(\bar{a}) \cdot n)$ [[exercise]]

Prop: $\Pr[D(\bar{a}) \ge k \cdot \log n] \le \frac{1}{n^{\Theta(c)}}$

Pf: focus on element a_i
  define intervals $I_{i,1} \supseteq I_{i,2} \supseteq \cdots \supseteq \emptyset$ (once a_i is pivot) by $I_{i,j}$ = { elements in the interval containing i at recursion depth j unpivoted }

eg



$[n] = I_{i,0}$
$I_{i,1}$
$I_{i,2} = \emptyset$

lem: $\mathbb{E}[|I_{i,j+1}| \mid |I_{i,j}| = k] \le \frac{7}{8} \cdot k$

Pf: if $I_{i,j} = \emptyset \Rightarrow I_{i,j+1} = \emptyset$

  else: w/p $\frac{1}{2}$ we split $I_{i,j}$ into a $(\frac{1}{4}, \frac{3}{4})$ balanced partition

  $\Rightarrow \mathbb{E}[\;'' \mid ''\;] \le \frac{1}{2} \cdot \frac{3}{4} \cdot |k| + \frac{1}{2} \cdot |k| = \frac{7}{8} |k|$

Cor: $\mathbb{E}[|I_{i,j}|] \le \left(\frac{7}{8}\right)^j \cdot n$

Pf. $\mathbb{E}[|I_{i,j}|] = \mathbb{E}_k\left[\underbrace{\mathbb{E}[|I_{i,j}| \mid |I_{i,j-1}| = k]}_{\le 7/8 \cdot k}\right] \le 7/8\, \mathbb{E}[|I_{i,j-1}|]$
$\le \cdots$
$\le (7/8)^j \cdot n$

$\Pr[D(\bar{a}) \ge j] = \Pr[\exists i \; |I_{i,j}| \ge 1] \overset{\text{union}}{\le} n \cdot \Pr[I_{1,j} \ge 1]$
$\le (7/8)^j \cdot n^2$

Cor: $\Pr[T(\bar{a}) \ge c \cdot n \log n] \le \frac{1}{n^{\Theta(c)}}$ $\le \frac{1}{n^{\Theta(c)}}$ if $j = c \log n$

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logistics: pset 3 due W10
  midterm 1 on Oct 7

today: - quicksort
  - concentration bounds

next time: - limited independence
  - hashing