cs473: Algorithms

Lecture 5: Dynamic Programming

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Overview

logistics:

- pset1 out, due W10 (tomorrow) — can submit in *groups* of ≤ 3

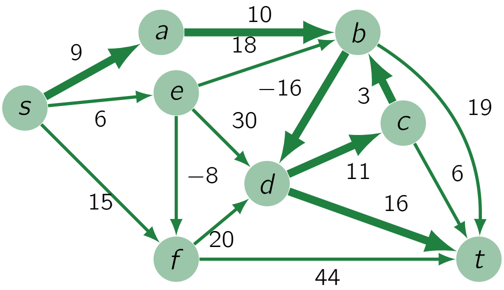
last time:

- dynamic programming on trees
- maximum independent set
- dominating set

today:

- shortest paths
  - with negative lengths
  - all-pairs

Shortest Paths, with Negative Lengths



total cost:

$$9 + 10 + (-16 + 11 + 3) \cdot k + (-16) + 16$$
$$= 19 - 3k \to -\infty$$

questions:

- what is the length of the shortest path between s and t?
- what is the length of the shortest path from s to *every* other node?
- what happens if we get lost?
- how to deal with *negative cycles*?

remarks:

- computing the length of the shortest *simple* $s \rightsquigarrow t$ path (with possibly negative lengths) is NP-hard — contains the *Hamiltonian path* problem

Shortest Paths, with Negative Lengths (II)

Definition

$G = (V, E)$ *directed* (simple) graph, with edge length function $\ell : E \to \mathbb{Z}$.

- A **path in** G is a sequence of *distinct* vertices $v_0, v_1, \dots, v_k \in V$ such that $(v_i, v_{i+1}) \in E$ for all i. An (s,t)-path is a path where $v_0 = s$ and $v_k = t$.
- A **walk in** G is a sequence of vertices $v_0, v_1, \dots, v_k \in V$ such that $(v_i, v_{i+1}) \in E$ for all i. An (s,t)-walk is a walk where $v_0 = s$ and $v_k = t$.
- The **length of a walk** is the sum of the edge lengths $\sum_i \ell(v_i, v_{i+1})$.
- The **distance from** s **to** t **in** G, denoted $\text{dist}(s,t)$, is the length of the shortest (s,t)-walk, $\text{dist}(s,t) := \min_{(s,t)\text{-walk } w} \ell(w)$.

remarks:

- if (s,t)-walk containing a negative length cycle $\implies$ $\text{dist}(s,t) = -\infty$
- if *no* (s,t)-walk containing a negative length cycle $\implies$ shortest walk is a *path* $\implies$ shortest walk $\leq n-1$ edges and is of finite length

Shortest Paths, with Negative Lengths (III)

Definition

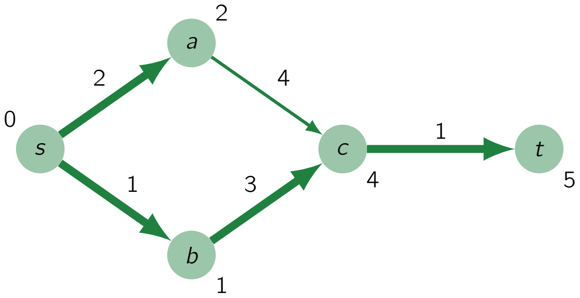
$G = (V, E)$ directed (simple) graph, with edge length function $\ell : E \to \mathbb{Z}$. The **(single-source) shortest path problem (with negative weights)** is to:

- given $s, t \in V$, find a minimum length (s, t)-path or find an (s, t)-walk with a negative cycle ($\implies$ $\text{dist}(s, t) = -\infty$)
- given $s \in V$, compute $\text{dist}(s, t)$ for all $t \in V$
- determine if G has *any* negative cycle

remarks:

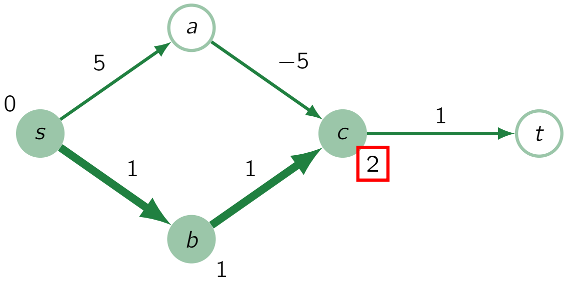
- negative lengths can be natural in modelling real life
  - e.g., demand/supply on an electrical grid
  - negative cycles manifest as *arbitrage*
- negative lengths can arise as by-products of other algorithms, e.g., flows in graphs

Dijkstra's algorithm: greedily grow shortest paths from source s



Dijkstra's Algorithm, with Negative Lengths?

Dijkstra's algorithm: greedily grow shortest paths from source s



remarks:

- greedy exploration, ordering vertices $v \in V$ by $\mathrm{dist}(s, v)$ — without updates!

$\Longrightarrow$ algorithm assumes the distance only grows as the graph is explored

$\equiv$ assumes all edge lengths are non-negative

Shortest Paths, with Negative Lengths (IV)

Lemma

$G = (V, E)$ directed (simple) graph, with edge length function $\ell : E \to \mathbb{Z}$. If $s = v_0 \to v_1 \to v_2 \to \cdots \to v_k = t$ is a shortest (s, t)-walk, then

1. *$s \to v_1 \to \cdots \to v_i$ is a shortest (s, v_i)-walk, for $i \leq k$*
2. **if** *ℓ is non-negative,* $\text{dist}(s, v_i) \leq \text{dist}(s, v_{i+1})$ *for all i*

Proof.

(1) Cut and paste. (2) Clear. □

remarks:

- shortest walks *are* shortest paths, if no negative cycle
- Dijkstra's algorithm defines subproblems by restricting the graph by $\text{dist}(s, \cdot)$
- *idea:* parameterize subproblems by *number* of edges in a walk, *and* allow updates to $\text{dist}(s, \cdot)$

Shortest Paths, with Negative Lengths (V)

Definition

$G = (V, E)$ directed (simple) graph, with edge length function $\ell : E \to \mathbb{Z}$. For $s, t \in V$, define $\mathrm{dist}_k(s, t)$ to be the length of the shortest (s, t)-walk *using* $\leq k$ *edges*.

$$\mathrm{dist}_k(s, t) := \min_{\substack{(s,t)\text{-walk } w \\ |w| \leq k}} \ell(w) .$$

remarks:

- $\mathrm{dist}_k(s, t) = \infty$ if no $(\leq k)$-edge (s, t)-walk
- $\mathrm{dist}_0(s, s) = 0$, $\mathrm{dist}_0(s, v) = \infty$ for $v \neq s$

Shortest Paths, with Negative Lengths (VI)

Lemma

$G = (V, E)$, $\ell : E \to \mathbb{Z}$. *Then for all* $s, t \in V$,

$$\text{dist}_k(s,t) = \min \begin{cases} \text{dist}_{k-1}(s,t) \\ \min_{v \in V} \{\text{dist}_{k-1}(s,v) + \ell(v,t)\} \end{cases} .$$

Proof.

Let $s = v_0 \to v_1 \to v_2 \to \cdots \to v_j = t$ be a shortest length $j \le k$ (s,t)-walk. Then,

- $j < k$: hence this is a $(\le k-1)$-edge (s,t)-walk of length $\text{dist}_{k-1}(s,t)$
- $j = k$: hence $s = v_0 \to v_1 \to v_2 \to \cdots \to v_{k-1}$ is a shortest length $(\le k-1)$-edge (s, v_{k-1}) walk $\implies$ can add $\ell(v_{k-1}, t)$ to reach t □

remark: $\ell(v,t) = \infty$ if there is no edge

Shortest Paths, with Negative Lengths (VII)

Theorem

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, $s \in V$, *with every vertex reachable from* s.

1. *If there are no negative length cycles, then for all* $v \in V$, $\text{dist}_{n-1}(s, v) \leq \text{dist}_n(s, v)$, *and even* $\text{dist}_{n-1}(s, v) = \text{dist}(s, v)$.
2. *If for all* $v \in V$, $\text{dist}_{n-1}(s, v) \leq \text{dist}_n(s, v)$, *then there are no negative length cycles.*

Shortest Paths, with Negative Lengths (VIII)

Lemma

$G = (V, E)$, $\ell : E \to \mathbb{Z}$. Then for all $s, t \in V$,

$$\mathrm{dist}_k(s,t) = \min \begin{cases} \mathrm{dist}_{k-1}(s,t) \\ \min_{v \in V}\{\mathrm{dist}_{k-1}(s,v) + \ell(v,t)\} \end{cases} .$$

Corollary

For all $k \geq 0$,

- $\mathrm{dist}_k(s,t) \leq \mathrm{dist}_{k-1}(s,t)$
- *If for all $v \in V$,* $\mathrm{dist}_k(s,t) = \mathrm{dist}_{k-1}(s,t)$

$\implies$ *for all $v \in V$,* $\mathrm{dist}_{k+1}(s,t) = \mathrm{dist}_k(s,t)$

$\implies$ *for all $v \in V$,* $\mathrm{dist}_{k+2}(s,t) = \mathrm{dist}_{k+1}(s,t) \implies \cdots$

Shortest Paths, with Negative Lengths (IX)

Proposition

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, $s \in V$, with every vertex reachable from s. If there are no negative length cycles, then for all $v \in V$, $\text{dist}_{n-1}(s, v) \leq \text{dist}_n(s, v)$.

Proof.

Let $s = v_0 \to v_1 \to \cdots \to v_{k-1} \to v_k = v$ be a walk of $(\leq n)$-edges, with length $\text{dist}_n(s, v)$.

- If $k < n$, then this is a $(< n)$-edge walk and hence of length $\geq \text{dist}_{n-1}(s, v)$.
- If $k = n$, then the walk visits $n + 1$ vertices $\implies$ some vertex is repeated $\equiv$ there is a cycle. As the cycle is of non-negative length $C \geq 0$, we can remove it to obtain a $(< n)$-edge (s, v)-walk of value $d = \text{dist}_n(s, v) - C$ with $\text{dist}_n(s, v) \geq d \geq \text{dist}_{n-1}(s, v)$. □

Shortest Paths, with Negative Lengths (X)

Proposition

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, $s \in V$, with every vertex reachable from s. If for all $v \in V$, $\text{dist}_{n-1}(s, v) \leq \text{dist}_n(s, v)$, then $\lim_{k\to\infty} \text{dist}_k(s, v)$ is finite for all $v \in V$.

Proof.

By previous corollary, for all $v \in V$, $\text{dist}_{n-1}(s, v) \geq \text{dist}_n(s, v) \implies$ for all $v \in V$, $\text{dist}_{n-1}(s, v) = \text{dist}_n(s, v) = \text{dist}_{n+1}(s, v) = \text{dist}_{n+2}(s, v) = \cdots$. As all v are reachable from $s \implies \text{dist}_{n-1}(s, v) \leq \infty$ for all k and v. Hence $\lim_{k\to\infty} \text{dist}_k(s, v) = \text{dist}_{n-1}(s, v)$ is finite for all v. □

Shortest Paths, with Negative Lengths (XI)

Proposition

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, $s \in V$, with every vertex reachable from s. If there is a (s, v)-walk containing a negative length cycle, then $\lim_{k\to\infty} \mathrm{dist}_k(s, v) = -\infty$.

Proof.

Let $s \rightsquigarrow u \rightsquigarrow u \rightsquigarrow v$ be an (s, v)-walk with length L, where $u \rightsquigarrow u$ is a negative length cycle of length $-C < 0$. Then consider the (s, v)-walk $s \rightsquigarrow u \rightsquigarrow u \rightsquigarrow u \rightsquigarrow v$, which is of value $L - C$. Hence, for any j there is (s, v)-walk of length $L - C \cdot j$. Hence $\lim_{k\to\infty} \mathrm{dist}_k(s, v) = -\infty$. □

Shortest Paths, with Negative Lengths (XII)

Proposition

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, $s \in V$, *with every vertex reachable from* s. *If for all* $v \in V$, $\text{dist}_{n-1}(s, v) \leq \text{dist}_n(s, v)$, $\lim_{k\to\infty} \text{dist}_k(s, v)$ *is finite for all* $v \in V$.

Proposition

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, $s \in V$, *with every vertex reachable from* s. *If there is a* (s, v)*-walk containing a negative length cycle, then* $\lim_{k\to\infty} \text{dist}_k(s, v) = -\infty$.

Corollary

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, $s \in V$, *with every vertex reachable from* s. *If for all* $v \in V$, $\text{dist}_{n-1}(s, v) \leq \text{dist}_n(s, v)$, *then there are no negative length cycles.*

Shortest Paths, with Negative Lengths (VII)

Theorem

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, $s \in V$, with every vertex reachable from s.

1. *If there are no negative length cycles, then for all $v \in V$,* $\mathrm{dist}_{n-1}(s, v) \le \mathrm{dist}_n(s, v)$, *and* $\mathrm{dist}_{n-1}(s, v) = \lim_{k\to\infty} \mathrm{dist}_k(s, v) = \mathrm{dist}(s, v)$.
2. *If for all $v \in V$,* $\mathrm{dist}_{n-1}(s, v) \le \mathrm{dist}_n(s, v)$, *then there are no negative length cycles.*

Bellman-Ford

(single source) shortest paths: source $s \in V$, can reach every other node

```
for each v ∈ V
    d_0[s][v] = ∞
d_0[s][s] = 0
for 1 ≤ k ≤ n,  v ∈ V
    d_k[s][v] = d_{k−1}[s][v]
    for u ∈ N^−(v)
        d_k[s][v] = min{d_k[s][v], d_{k−1}[s][u] + ℓ(u, v)}
for v ∈ V
    if d_n[s][v] < d_{n−1}[s][v]
        return ‘‘negative cycle detected’’
return d_{n−1}[s][·]
```

correctness: clear

complexity:

- time
  - clearly $O(n^3)$
  - *better:* $O(mn)$, $d_k[s][\cdot]$ updates along edges
- space
  - clearly $O(n^2)$
  - *better:* only store $d_{\text{cur}}[s][\cdot]$ and $d_{\text{prev}}[s][\cdot] \implies O(n)$

Bellman-Ford (II)

remarks:

- compute actual paths by storing pointers indicating *how* $d_k[s][\cdot]$ was updated, e.g.,
$$v_{k-1} = \arg\min_{u \in V} \{\mathrm{dist}_{k-1}(s,u) + \ell(u, v_k)\} \; .$$
- detecting negative cycles
  - Bellman-Ford will detect any negative cycles *reachable from s* in G

  $\implies$ one Bellman-Ford call *per vertex* will detect if there is *any* negative cycle in G $\implies$ $O(mn^2)$ time
  - *better:* consider $G' = (V \cup \{s'\}, E \cup \{(s', v)\}_{v \in V})$ with $\ell'(s', v) = 0$

  $\implies$ *all* negative cycles in G are reachable from s' in G'

  $\implies$ *one* Bellman-Ford required $\implies$ $O(mn)$ time
- directed *acyclic* graphs
  - no (negative) cycles
  - can simplify Bellman-Ford so $\mathrm{dist}_k(s, \cdot)$ only updates v_k, according to topological ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ — yields Dijkstra-esque algorithm

  $\implies$ $O(m + n)$ time (**exercise**)

All-Pairs Shortest Paths

Definition

$G = (V, E)$ directed (simple) graph, $\ell : E \to \mathbb{Z}$. The **shortest path problem** is to:

- given $s, t \in V$, find a minimum length (s, t)-path
- given $s \in V$, compute $\text{dist}(s, t)$ for all $t \in V$ (single-source)
- compute $\text{dist}(s, t)$ for all $s, t \in V$ (all pairs)

single-source:

- *Dijkstra:*
  - non-negative lengths
  - $O((m + n) \log n)$ time (heaps), $O(m + n \log n)$ (Fibonacci heaps)
- *Bellman-Ford:*
  - arbitrary weights
  - $O(mn)$ time

All-Pairs Shortest Paths (II)

Definition

$G = (V, E)$ directed (simple) graph, $\ell : E \to \mathbb{Z}$. The **shortest path problem** is to:

- given $s, t \in V$, find a minimum length (s, t)-path
- given $s \in V$, compute $\text{dist}(s, t)$ for all $t \in V$ (single-source)
- compute $\text{dist}(s, t)$ for all $s, t \in V$ (all pairs)

all-pairs:

- n runs of *Dijkstra:*
  - non-negative lengths
  - $O(n \cdot (m + n) \log n)$ time (heaps), $O(n \cdot (m + n \log n))$ (Fibonacci heaps)
- n runs of *Bellman-Ford:*
  - arbitrary weights
  - $O(n \cdot mn)$ time $\mapsto \Theta(n^4)$ if $m = \Theta(n^2)$

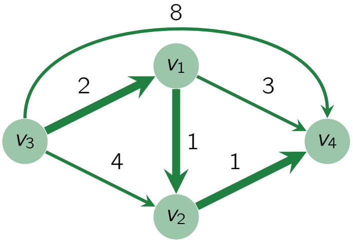
question: can we do better?

All-Pairs Shortest Paths (III)

idea: use a new parameterization of the subproblems

Definition

$G = (V, E)$ directed (simple) graph, with edge length function $\ell : E \to \mathbb{Z}$. Order V as $v_1 \prec v_2 \prec \cdots \prec v_n$. A (u, v)-walk $u = w_0 \to w_1 \to \cdots \to w_i = v$ has **intermediate index** $\le j$, if $w_1, \ldots, w_{i-1} \in \{v_1, \ldots, v_j\}$. For $s, t \in V$, define $\text{dist}^k(s, t)$ to be the length of the shortest (s, t)-walk of intermediate index $\le k$.



- $\text{dist}^0(v_3, v_4) = \ell(v_3, v_4) = 8$
- $\text{dist}^1(v_3, v_4) = 5$
- $\text{dist}^2(v_3, v_4) = 4$

All-Pairs Shortest Paths (IV)

Lemma

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, with no negative cycles. Then for all $s, t \in V$, $\text{dist}^0(s, t) = \ell(s, t)$, and

$$\text{dist}^k(s, t) = \min \begin{cases} \text{dist}^{k-1}(s, t) \\ \text{dist}^{k-1}(s, v_k) + \text{dist}^{k-1}(v_k, t) \end{cases} .$$

Proof.

Let $s = w_0 \to w_1 \to w_2 \to \cdots \to w_i = t$ be a shortest length (s, t)-walk of intermediate index $\leq k$ and length $\text{dist}^k(s, t)$. There are two cases:

- index $< k$: hence is of value $\text{dist}^{k-1}(s, t)$
- index $= k$:
  - no negative cycles $\implies$ shortest *walk* is *path* $\implies$ v_k appears exactly once

$\implies$ $s \rightsquigarrow v_k$ path and $v_k \rightsquigarrow t$ path are of index $< k$, and must be *shortest* paths □

Floyd-Warshall

```
for 1 ≤ i,j ≤ n
    d^0[i][j] = ℓ(i,j)
for 1 ≤ k ≤ n
    for 1 ≤ i,j ≤ n
        d^k[i][j] = min { d^{k-1}[i][j]
                        { d^{k-1}[i][k] + d^{k-1}[k][j]
for 1 ≤ i ≤ n
    if d^n[i][i] < 0
        return ‘‘negative cycle detected’’
```

$$d^k[i][j] = \min \begin{cases} d^{k-1}[i][j] \\ d^{k-1}[i][k] + d^{k-1}[k][j] \end{cases}$$

remarks:

- compute actual paths by storing pointers indicating *how* $d^k[\cdot][\cdot]$ was updated

complexity:

- $O(n^3)$ time
- space
  - clearly $O(n^3)$
  - *better:* only store $d^{\text{cur}}[\cdot][\cdot]$ and $d^{\text{prev}}[\cdot][\cdot] \implies O(n^2)$

correctness:

- if *no* negative cycles, correctness is clear
- if *some* negative cycle, ???

Floyd-Warshall (II)

Proposition

$G = (V, E)$, $\ell : E \to \mathbb{Z}$, with some negative cycle. Then the Floyd-Warshall algorithm correctly detects this cycle.

Proof.

Let $k \leq n$ be the minimum index of a negative length cycle $k = \min_{\text{negative length } C} \max_{i : v_i \in C} i$. Pick such a cycle C, where C is $v_k = w_0 \to w_1 = v_i \to \cdots \to w_j = v_k$. By choice of k,

- $d^{k-1}[k][i] = \text{dist}^{k-1}(k, i) \leq \ell(w_0, w_1)$
- $d^{k-1}[i][k] = \text{dist}^{k-1}(i, k) \leq \ell(w_1, w_2) + \cdots + \ell(w_{j-1}, w_j)$

$\implies$ $d^k[k][k] \leq d^{k-1}[k][i] + d^{k-1}[i][k] = \ell(w_0, w_1) + \cdots + \ell(w_{j-1}, w_j) = \ell(C) < 0$

$\implies$ $d^{k+1}[k][k] \leq d^k[k][k] < 0$

$\implies$ $d^n[k][k] < 0 \implies$ negative cycle detected □

Floyd-Warshall

```
for 1 ≤ i,j ≤ n
    d⁰[i][j] = ℓ(i,j)
for 1 ≤ k ≤ n
    for 1 ≤ i,j ≤ n
        dᵏ[i][j] = min { dᵏ⁻¹[i][j] ; dᵏ⁻¹[i][k] + dᵏ⁻¹[k][j] }
for 1 ≤ i ≤ n
    if dⁿ[i][i] < 0
        return ‘‘negative cycle detected’’
```

$$d^k[i][j] = \min \begin{cases} d^{k-1}[i][j] \\ d^{k-1}[i][k] + d^{k-1}[k][j] \end{cases}$$

remarks:

- compute actual paths by storing pointers indicating *how* $d^k[\cdot][\cdot]$ was updated

complexity:

- $O(n^3)$ time
- space
  - clearly $O(n^3)$
  - *better:* only store $d^{\text{cur}}[\cdot][\cdot]$ and $d^{\text{prev}}[\cdot][\cdot] \implies O(n^2)$

correctness:

- if *no* negative cycles, correctness is clear
- if *some* negative cycle, correctness is now done

Overview (II)

logistics:

- pset1 out, due W10 (tomorrow) — can submit in *groups* of ≤ 3

today:

- shortest paths
  - with negative lengths — Bellman-Ford in $O(mn)$ time
  - all-pairs — Floyd-Warshall in $O(n^3)$ time

next time:

- *more* dynamic programming

TOC