cs473: Algorithms

Lecture 4: Dynamic Programming

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Overview

logistics:

- pset1 out, due W10 (next week) — can submit in *groups* of ≤ 3
- if you are waiting to enroll: post private note in piazza with name, netid, major by *today* — we have a limited number of additional spots in the online section and will prioritize enrollment

last time:

- recursion, memoization, dynamic programming
- fibonacci numbers, edit distance, knapsack

today:

- dynamic programming *on trees*
- maximum independent set
- dominating set

Dynamic Programming

dynamic programming:

- develop recursive algorithm
- understand structure of subproblems
  - names of subproblems
  - number of subproblems
  - dependency graph amongst subproblems
- memoize (implicitly, or explicitly)
- analysis (time, space)
- further optimization

remarks:

- memoizing a recursive algorithm does not necessarily lead to an efficient algorithm (e.g., knapsack problem) — you need the *right* recursion
- recognizing that dynamic programming applies to a problem can be non-obvious

Trees

fact:

- many computational problems ask to optimize an objective over a graph
- many graph optimization problems are NP-hard
- *yet*: many NP-hard graph optimization problems can be efficiently solved when the graph is a *tree*

remarks:

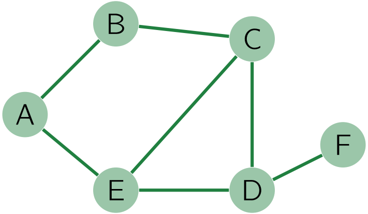
- dynamic programming over graphs often relies on decomposing the graph into subgraphs, but there are many subgraphs and they relate to each other in complicated ways
- trees can be easily decomposed into sub-trees, which are easily related to each other $\implies$ trees are amenable to divide and conquer, and dynamic programming more generally
- dynamic programming on trees often generalizes to graphs that have low *treewidth*

Maximum Independent Set

Definition

Let $G = (V, E)$ be an undirected (simple) graph. An **independent set of** G is a subset $S \subseteq V$ such that there are no edges in G between vertices in S. That is, for all $u, v \in S$ that $(u, v) \notin E$.

ex:



Independent sets include $\emptyset$, $\{A, C\}$, and $\{B, E, F\}$.

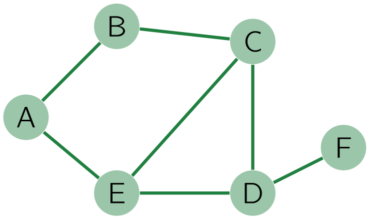
Maximum Independent Set (II)

Definition

The **maximum independent set (MIS)** problem is to, given a undirected (simple) graph $G = (V, E)$ output the size of the largest independent set in G. That is, output

$$\alpha(G) := \max_{S \subseteq V, S \text{ independent set of } G} |S| .$$

ex:



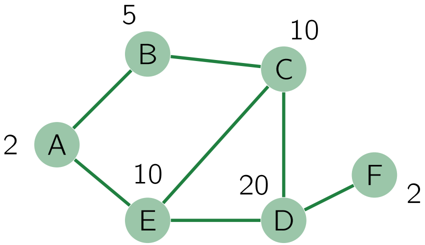
$\alpha(G) = 3$

Maximum Independent Set (III)

Definition

The **maximum weight independent set** problem is to, given a undirected (simple) graph $G = (V, E)$ *and* a weight function $w : V \to \mathbb{N}$, output the weight of the maximum weight independent set in G. That is, output

$$\max_{\substack{S \subseteq V \\ S \text{ independent set of } G}} \sum_{v \in S} w(v) .$$



Maximum Independent Set (IV)

remarks:

- maximum (weight) independent set (MIS) is solvable via brute force: try *all* possible subsets $\implies$ solvable in time $O(n^{O(1)} 2^n)$
- no efficient algorithm *currently* known
- MIS is NP-hard $\implies$ an efficient algorithm *not* expected to exist
- MIS is efficiently solvable if the underlying graph is a *tree*

Maximum Independent Set (V)

For vertex v, let $N(v)$ denote the subset $S \subseteq V$ of *neighbors* of v.

Lemma

$G = (V, E)$, $w : V \to \mathbb{N}$. *Then for any* $v \in V$,

$$\mathrm{MIS}(G) = \max\left\{\mathrm{MIS}(G - v), \mathrm{MIS}(G - v - N(v)) + w(v)\right\}.$$

Proof.

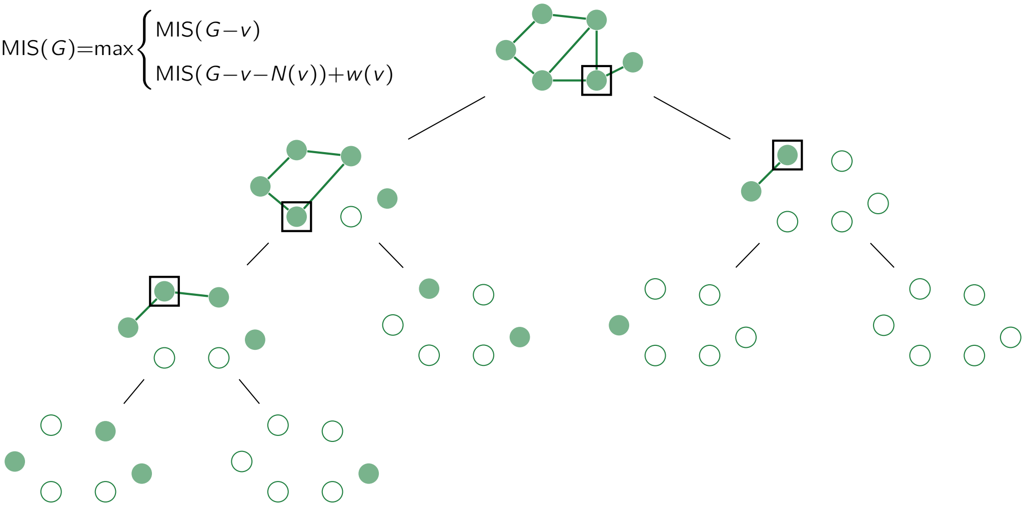
For any set S independent in G, either $v \notin S$ or $v \in S$.

- $G - v$: any set $T \subseteq V \setminus \{v\}$ independent in $G - v$ has $T \subseteq V$ independent in G
- $G - v - N(v)$: any set $T \subseteq V \setminus (\{v\} \cup N(v))$ independent in $G - v - N(v)$ has $T \cup \{v\} \subseteq V$ independent in G

Any set S independent in G must be of the above two cases. Now maximize. □

Maximum Independent Set (VI)

$$\text{MIS}(G) = \max \begin{cases} \text{MIS}(G - v) \\ \text{MIS}(G - v - N(v)) + w(v) \end{cases}$$



Maximum Independent Set (VII)

recursive-MIS$(G = (V, E))$:
```
    if V = ∅
        return 0
    choose v ∈ V
    return max(recursive-MIS(G − v), recursive-MIS(G − v − N(v)) + w(v))
```

correctness: clear

complexity: $n := |V|$

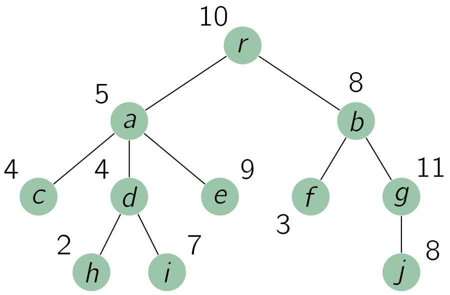
- $T(0), T(1) \geq \Omega(1)$. $T(n) \geq T(n-1) + T(n-1-\deg(v))$
- silly case: G has no edges $\implies$ for all v, $\deg(v) = 0$

$\implies$ $T(n) \geq 2T(n-1) \geq 4T(n-2) \geq \cdots \geq 2^n \cdot T(1) \geq \Omega(2^n)$.

- when G has no edges then clearly $\mathrm{MIS}(G) = |V|$, but this worst-case runtime is hard to avoid
- memoization does not obviously help — subproblems correspond to subgraphs, of which there are possibly exponentially many

Maximum Independent Set, in Trees

question: maximum weight independent set, in trees?

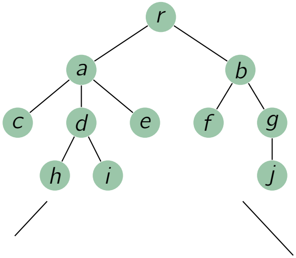


question:

- how to bound the number of subproblems in recursive algorithm?
- how to pick which vertex $v \in V$ to eliminate?

Maximum Independent Set, in Trees (II)

$$\mathrm{MIS}(G) = \max \begin{cases} \mathrm{MIS}(G - v) \\ \mathrm{MIS}(G - v - N(v)) + w(v) \end{cases}$$



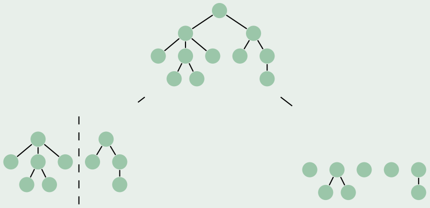
Maximum Independent Set, in Trees (III)

Lemma

Let $T = (V, E)$ be a tree, with **root** *$v \in V$. Then*

- *$T - v$ is a forest, with each tree associated to a child u of v.*
- *$T - v - N(v)$ is a forest, with each tree associated to a grandchild w of v.*

Proof.



□

Maximum Independent Set, in Trees (III)

Lemma

Let $T = (V, E)$ be a tree, with **root** $v \in V$. *Then*

- $T - v$ *is a forest, with each tree associated to a child* u *of* v.
- $T - v - N(v)$ *is a forest, with each tree associated to a grandchild* w *of* v.

Corollary

Let $T = (V, E)$ be a tree. Pick a root $r \in V$ for T to create the rooted tree (T, r). If you run **recursive-MIS** *on T and always eliminate the nodes who were closest to r in T, then the result subproblems exactly correspond to rooted subtrees of (T, r)*

$\implies$ $\leq |V|$ *subproblems*

$\implies$ *memoized recursive algorithm is efficient*

Maximum Independent Set, in Trees (IV)

For a rooted tree T with root r, for $v \in V$ define $T(v)$ to be the subtree of T descending from v. The recursive formula is then:

$$\text{MIS}(T) = \max \begin{cases} \sum_{v \in N(v)} \text{MIS}(T(v)) \\ \left(\sum_{v \in N(N(v))} \text{MIS}(T(v))\right) + w(v) \end{cases}$$

dependency graph:

- subproblems are rooted subtrees of (T, r)
- a subtree $T(v)$ depends on all of subtrees $T(u)$ where u is a descendent of v

$\Longrightarrow$ iterating over V in post-order traversal of T will satisfy the dependency graph

Maximum Independent Set, in Trees (V)

iterative algorithm:

iter-MIS-tree$(T = (V, E))$:

```
    let v_1, v_2, ..., v_n be a post-order traversal of nodes of T
        ==> v_n is the root
    for 1 <= i <= n
```

$$M[i] = \max \begin{cases} \sum_{j: v_j \in N(v_i)} M[j] \\ \left(\sum_{j: v_j \in N(N(v_i))} M[j] \right) + w(v_i) \end{cases}$$

```
    return M[n]
```

correctness: clear

complexity:

- $O(n)$ space to store $M[\cdot]$
- time
  - *naive:* $O(n)$ time per node, n nodes $\implies O(n^2)$
  - *better:* each node v_j has its $M[j]$ value read by parent, and by grandparent $\implies$ $O(1)$ work per n nodes $\implies O(n)$ time

Dynamic Programming, in Trees

question: why does dynamic programming work on trees?

Definition

$G = (V, E)$. A set of nodes $S \subseteq V$ is a **separator for** G if $G - S$ has at ≥ 2 connected components, that is, $G - S$ is disconnected.
S is a **balanced** if each connected component of $G - S$ has $\leq \frac{2}{3} \cdot |V|$ vertices.

e.g., in trees, *every* vertex is a separator, but not all are *balanced*.
remarks:

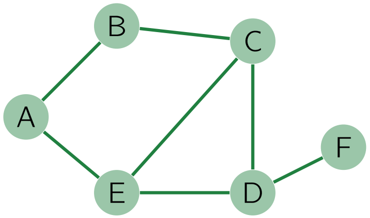
- every tree T has a balanced separator consisting of a single node
- dynamic-programming + small balanced separators $\implies 2^{O(\sqrt{n})}$-time MIS algorithm for planar graphs

Minimum Dominating Set

Definition

Let $G = (V, E)$ be an undirected (simple) graph. A **dominating set of** G is a subset $S \subseteq V$ such that for all $v \in V$, either $v \in S$, or v has neighbor $u \in N(v)$ with $u \in S$.

ex:



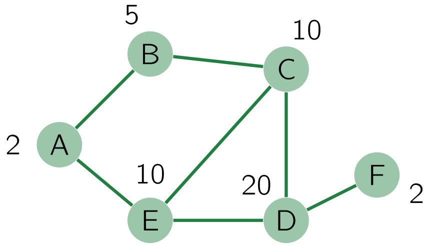
Dominating sets include $\{A, B, C, D, E, F\}$, $\{E, C, F\}$, and $\{A, B, F\}$.

Minimum Dominating Set (II)

Definition

The **minimum weight dominating set** problem is to, given a undirected (simple) graph $G = (V, E)$ *and* a weight function $w : V \to \mathbb{N}$, output the weight of the minimum weight dominating set in G. That is, output

$$\max_{\substack{S \subseteq V \\ S \text{ dominating set of } G}} \sum_{v \in S} w(v) \, .$$



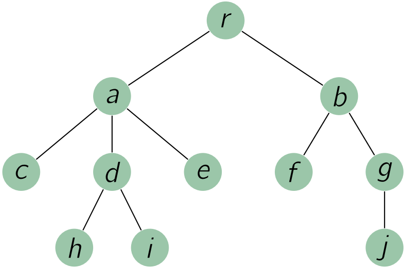
Minimum Dominating Set (III)

remarks:

- minimum (weight) dominating set is solvable via brute force: try *all* possible subsets $\implies$ solvable in time $O(n^{O(1)}2^n)$
- no efficient algorithm *currently* known
- minimum weight dominating set is NP-hard $\implies$ an efficient algorithm *not* expected to exist
- minimum weight dominating set is efficiently solvable if the underlying graph is a *tree*

Minimum Dominating Set, in Trees

question: copy&paste from MIS on trees?



Let $T(v)$ denote the subtree rooted at $v \in V$, and let $S(v)$ be any minimum weight dominating set for $T(v)$.

building $S(r)$**:**

- $r \in S$:
  - could take any $S(a) \cup S(b) \cup \{r\}$
  - *but can better:* if we cover r then a, b do not need to be covered — only need a "mostly" dominating set on $T(a)$ and $T(b)$
- $r \notin S$:
  - could try to take any $S(a) \cup S(b)$, but how to dominate r?
  - need a "extra" dominating set from *one* of $T(a)$ and $T(b)$

question: how to parameterize these subproblems?

Minimum Dominating Set, in Trees (II)

Definition

Let $T = (V, E)$ be a rooted tree with root r.

- A **type-0** dominating set for T is an actual dominating set.
- A **type-1** dominating set for T is an actual dominating set S *where* $r \in S$.
- A **type-2** dominating set for T is a subset $S \subseteq V$ such that for all $v \in V \setminus \{r\}$, either $v \in S$ or v has a neighbor $u \in N(v)$ with $u \in S$.

For $b \in \{0, 1, 2\}$, define OPT_b to be the minimum weight dominating set for T of b-type. Define $\mathrm{OPT}_b(v)$ to be the OPT_b for the subtree of T rooted at v.

base case:

- T has no vertices $\implies \mathrm{OPT}_b(T) = 0$
- extends gracefully by the following conventions:
  - for $S = \emptyset$, $\sum_{v \in S} f(v) = 0$
  - for $S = \emptyset$, $\min_{v \in S} f(v) = \infty$

Minimum Dominating Set, in Trees (III)

T rooted tree with root r. $T(v)$ is subtree rooted at v.

- **type-0**: regular dominating set
- **type-1**: dominating set which includes root r
- **type-2**: dominating set which is relaxed at root r

Lemma

$$\mathrm{OPT}_0(r) = \min \begin{cases} \left(\sum_{v \in N(r)} \mathrm{OPT}_2(v)\right) + w(r) \\ \min_{v \in N(r)} \left(\mathrm{OPT}_1(v) + \sum_{u \in N(r) \setminus \{v\}} \mathrm{OPT}_0(u)\right) \end{cases} .$$

Proof.

- in optimum S, $r \in S$
- in optimum S, $r \notin S$ and r dominated by child $v \in S$ □

Minimum Dominating Set, in Trees (IV)

T rooted tree with root r. $T(v)$ is subtree rooted at v.

- **type-0**: regular dominating set
- **type-1**: dominating set which includes root r
- **type-2**: dominating set which is relaxed at root r

Lemma

$$\mathrm{OPT}_1(r) = \left(\sum_{v \in N(r)} \mathrm{OPT}_2(v) \right) + w(r)\ .$$

Proof.

In optimum S, $r \in S$. □

Minimum Dominating Set, in Trees (V)

T rooted tree with root r. $T(v)$ is subtree rooted at v.

- **type-0**: regular dominating set
- **type-1**: dominating set which includes root r
- **type-2**: dominating set which is relaxed at root r

Lemma

$$\mathrm{OPT}_2(r) = \min \begin{cases} \left(\sum_{v \in N(r)} \mathrm{OPT}_2(v)\right) + w(r) \\ \sum_{v \in N(r)} \mathrm{OPT}_0(v) \end{cases} .$$

Proof.

- in optimum S, $r \in S$
- in optimum S, $r \notin S$ and r does not need to be dominated by children □

Minimum Dominating Set, in Trees (VI)

T rooted tree with root r.

subproblems:

- **type-0**: regular dominating set
- **type-1**: dominating set which includes root r
- **type-2**: dominating set which is relaxed at root r

recursion:

- $\mathrm{OPT}_0(r)=\min \begin{cases} \left(\sum_{v\in N(r)} \mathrm{OPT}_2(v)\right)+w(r) \\ \min_{v\in N(r)} \left(\mathrm{OPT}_1(v)+\sum_{u\in N(r)\setminus\{v\}} \mathrm{OPT}_0(u)\right) \end{cases}$
- $\mathrm{OPT}_1(r)=\left(\sum_{v\in N(r)} \mathrm{OPT}_2(v)\right)+w(r)$
- $\mathrm{OPT}_2(r)=\min \begin{cases} \left(\sum_{v\in N(r)} \mathrm{OPT}_2(v)\right)+w(r) \\ \sum_{v\in N(r)} \mathrm{OPT}_0(v) \end{cases}$

$\mathrm{OPT}_0(r)$ is desired answer

recursive algorithm:

- $3\cdot n$ subproblems
- can implicitly memoize
- naively $O(n)$ work per node, can optimize to $O(n)$ total work as with MIS on trees

iterative algorithm:

- follow post-order traversal of rooted tree to satisfy dependencies
- optimize analysis to obtain $O(n)$ total work

details are an **exercise**

Dynamic Programming, in Trees (II)

remarks:

- dynamic program is about finding the *correct* recursion, and the correct recursion is intimately tied to understand the *structure* and *number* of subproblems
- trees can be easily decomposed into a (small) number of subtrees, this allows a small number of resulting subproblems
- dynamic programming on trees can often be generalized to graphs of small *treewidth*

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today:

- dynamic programming *on trees*
- maximum independent set
- dominating set

next time:

- *more* dynamic programming

TOC