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2019-10-31.4 $\rightleftarrows$ 2019-11-05.1
2019-11-05.2
cs473

cs473 Algorithms: Lecture 20

logistics:
- psets 8 due Thurs 10
- midterm 2 Nov 11 Monday - SC 1404, 7-9:30pm
  - Thurs is optional midterm review
  - syllabus, sample midterm, conflict exam → piazza

last time
- dual LP
- weak duality
- strong duality ⟹ max flow = min cut [arbitrary capacities]

today: strong duality

def. A *halfspace* is a set $\{x : \langle a, x\rangle \le b\} \subseteq \mathbb{R}^n$ ($a \in \mathbb{R}^n$, $b \in \mathbb{R}$)

A *polyhedron* is the intersection of a finite # of halfspaces

$$\{x : \underbrace{A}_{\mathbb{R}^{m\times n}} x \le \underbrace{b}_{\mathbb{R}^n}\} \subseteq \mathbb{R}^n$$

← not bounded

A *polytope* is a bounded polyhedron, ie $P \subseteq [-B, B]^n$ some B.

def: $\pi_{\le k} : \mathbb{R}^n \to \mathbb{R}^k$ is the *projection* onto the first k coordinates

$(x_1, \ldots, x_k, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$

Thm (Fourier Motzkin Elimination): polyhedron $P = \{x : Ax \le b\}$ $A \in \mathbb{R}^{m\times n}$ $b \in \mathbb{R}^m$ [drop last coord]

then
- $\pi_{<n}(P)$ is a polyhedron $P_{<n} = \{x_{<n} : Cx_{<n} \le d\}$ [projection of polyhedron is polyhedron] ($x_{<n} = (x_1, \ldots, x_{n-1})$)
- "$Cx_{<n} \le d$" is $\le m^2$ constraints on $n-1$ variables [complexity of projection]
- ↳ are non-negative linear combinations of "$Ax \le b$"

ie $C_{i,\circ} = (y_i^T A)_{<n}$ $\quad d_i = y_i^T b$, some $y_i \in \mathbb{R}^m_{\ge 0}$

w/ $(y_i^T A)_n = 0$

ie: $\langle C_{i,\circ}, x_{<n}\rangle = \langle y_i^T A, x\rangle$

$y_i^T \, A \, x \le y_i^T \, b$

$\square \, x \le \square$ [eliminated x_n from constraint]

$= \square \, x_{<n}$

thm (strong duality)

π: $\max \langle c, x\rangle$ st $Ax \le b$, $x \ge 0$

$\amalg$: $\min \langle b, y\rangle$ st $y^T A \ge c$, $y \ge 0$

if π feasible bounded ⟹ so is $\amalg$, $|\pi| = |\amalg|$

pf idea: $P = \{(z, x) : z - \langle c, x\rangle \le 0,\ Ax \le b,\ x \ge 0\}$ [z scalar]

$P_z = \pi_z(P) = \{z : z \le |\pi|\}$

if P_z polyhedron defined by eqns arising as non-negative linear combinations of eqns of P ⟺ "$z \le |\pi|$" is expressible by dual ⟹ dual provides optimal ub on primal → strong duality

[how to understand projections of polyhedra?]

idea: Gaussian Elimination

$$\begin{aligned} x + y &= 1 \\ -x + y &= 5 \\ 2x \quad &= 6 \end{aligned}$$

issues: cannot use negative coefficients → flip inequalities

pf: m eqns $\quad [m] = S_+ \sqcup S_0 \sqcup S_-$

$S_+ = \{i : A_{in} > 0\}$

S_0 $\quad =$

S_- $\quad <$

"$Cx_{<n} \le d$" is set of eqns
- $\{A_{i,\circ} x \le b_i\}_{i \in S_0}$ [did not involve x_n anyhow]
- $\left\{\underbrace{-A_{i,n}}_{\ge 0}(A_{j,\circ} x \le b_j) + \underbrace{A_{j,n}}_{\ge 0}(A_{i,\circ} x \le b_i)\right\}_{i \in S_-,\, j \in S_+}$

$= (A_{j,n} \cdot A_{i,\circ} - A_{i,n} A_{j,\circ})\, x \le A_{j,n} b_i - A_{i,n} b_j$

Clm: new eqns are non-neg linear combinations

pf: clear

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Clm: new eqns do not involve x_n

Pf: S_0 type: by def

$S_-|S_+$: $A_{i,n} \cdot A_{j,\cdot} x - A_{j,n} A_{i,\cdot} x \le \cdots$ (i, j)

$\text{coeff}_{x_n} = A_{i,n} \cdot A_{j,n} - A_{j,n} A_{i,n} = 0$

Clm: at most $|S_0| + |S_+| \cdot |S_-| \le (|S_0| + |S_-| + |S_+|)^2 = m^2$ eqns

Clm: $\pi_{<n}(P) \subseteq P_{<n} = \{x_{<n} : Cx_{<n} \le d\}$ [[easy direction]]

Pf: $x \in P \Rightarrow Ax \le b \Rightarrow y^T A x \le y^T b$ any $y \ge 0$ [[any derived eqn is valid on x]]

$\Rightarrow \underbrace{Cx}_{= Cx_{<n}} \le d \Rightarrow x_{<n} \in P_{<n}$

Clm: $P_{<n} \subseteq \pi_{<n}(P)$ [[hard direction]]

Pf: $x_{<n} \in P_{<n} \Leftrightarrow Cx_{<n} \le d$

want: $\exists x_n$ st $(x_{<n}, x_n) \in P \equiv \exists x_n \ \underbrace{A\begin{pmatrix} x_{<n} \\ x_n \end{pmatrix}}_{\equiv \underbrace{A_{\cdot,<n} x_{<n}}_{m \times (n-1)} + \underbrace{A_{\cdot,n} x_n}_{m \times 1}} \le b$

$\equiv \exists x_n \ \underbrace{A_{\cdot,n} x_n \le b - A_{\cdot,<n} x_{<n}}_{\{A_{i,n} x_n \le B_{i,n}\}_i}$

$[A_{\cdot,<n} \mid A_{\cdot,n}] [\ ] \le [\ ]$

subclaim: $i \in S_0 \Rightarrow A_{i,n} = 0$

$\Rightarrow \underbrace{A_{i,\cdot} x_{<n}}_{= A_{i,\cdot} x \text{ any } x_n} \le b$ [[$x_{<n} \in P_{<n}$]]

$\Rightarrow \forall x_n \ \underbrace{A_{i,n} x_n}_{=0} \le \underbrace{B_{i,n}}_{\ge 0}$ [[so can ignore these as they are satisfied]]

subclaim: $\{x_n : \forall i \ A_{i,n} x_n \le B_i\} = \{x_n : A_{i,n} x_n \le B_i, \ i \in S_+\} \equiv \{x_n : x_n \le \frac{B_i}{A_{i,n}}, i \in S_+\}$

$\cap \{ \quad " \quad i \in S_0\} = \mathbb{R}$

$\cap \{ \quad i \in S_-\} = \{x_n : x_n \ge \frac{B_i}{A_{i,n}} \ i \in S_-\}$ [[reverse inequality

$= \{x_n : \max_{i \in S_-} \frac{B_i}{A_{i,n}} \le x_n \le \min_{j \in S_+} \frac{B_j}{A_{j,n}}\}$

subclaim: is non empty [[exists coord to add to $x_{<n}$ to lift to P]]

PF: $\equiv \max_{i \in S_-} \frac{B_i}{A_{i,n}} \le \min_{j \in S_+} \frac{B_j}{A_{j,n}}$

$\equiv \forall i \in S_-, j \in S_+ \ \ \frac{B_i}{A_{i,n}} \le \frac{B_j}{A_{j,n}}$ mult by $\underbrace{A_{i,n}}_{<0} \underbrace{A_{j,n}}_{>0} < 0$

$\equiv A_{j,n} B_i \ge A_{i,n} B_j$

$\equiv A_{j,n}(b_i - A_{i,<n} x_{<n}) \ge A_{i,n}(b_j - A_{j,<n} x_{<n})$

$\equiv (A_{j,n} \cdot A_{i,<n} - A_{i,n} A_{j,<n}) x_{<n} \le A_{j,n} b_i - A_{i,n} b_j$ true, as equation of $Cx_{<n} \le d$

hence: $x_{<n} \in P_{<n} \Rightarrow Cx_{<n} \le d$

$\Rightarrow \forall i \in S_-, j \in S_+$

$\Rightarrow \max_{i \in S_-} \frac{B_i}{A_{i,n}} \le \min_{j \in S_+} \frac{B_j}{A_{j,n}} \Rightarrow \exists x_n \max " \le x_n \le \min "$

$\Rightarrow \{\exists x_n \ A_{i,n} x_n \le B_i, \forall i\}$ non empty

$\Rightarrow \exists x_n \ A_{\cdot,<n} x_{<n} + A_{\cdot,n} x_n \le b \Rightarrow x_{<n} \in \pi_{<n}(P)$ ∎

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Cor. polyhedron $P = \{x : Ax \le b\}$ A $m \times n$

then $\Pi_{\le k}(P)$ – is polyhedron $\{x_{\le k} : Cx_{\le k} \le d\}$

– $\leq m^{2^k}$ constraints [squares each step]

– "$Cx_{\le k} \le d$" are non-neg lin comb of $Ax \le b$

ie $C_{i,\cdot} = ((y_i^T) A)_{\le k}, d_i = (y_i^T) b$ $y_i \ge 0$ [non-neg of non-neg is non-neg]

Sketch: induction

w/ $(y_i^T A)$ has last $n-k$ entries as zero

$\Rightarrow$ Questions

$\langle C_{i,\cdot}, x_{\le k}\rangle = \langle (y_i^T A), x\rangle$

Cor: Π $\max \langle c, x\rangle$ st $Ax \le b$ [not canonical form]

define polyhedron P by $z - \langle c, x\rangle \le 0$, $Ax \le b$, P_z as projection of P eliminating x

then: Π unbounded $\equiv$ P_z has no constraints and then are the possibilities

Π infeasible $\equiv$ P_z defined by "$0 \le -1$"

Π feasible bounded $\equiv$ P_z ~~defined~~ by $z \le |\Pi|$

Pf. FM elim $\rightarrow$ $P_z = \{z \le |\Pi|\}$ defined by $\{\underbrace{\alpha_i}_{\ge 0 \text{ as FM uses non-neg combinations}} z \le \beta_i\}_i$
[empty if infeasible]
[all z if unbounded]

Clm: $\exists i$ $\alpha_i = 0, \beta_i < 0$ $\equiv$ $0 \cdot z < 0 \equiv 0 \le -1 \Rightarrow \Pi$ infeasible of $z - \langle c, x\rangle \le 0$

Clm. – can discard eqns w/ $\alpha_i = 0, \beta_i \ge 0$ $\equiv$ $0 \le \beta_i$ [vacuous]

– if no remaining eqns $\Rightarrow$ $P_1 = \mathbb{R} \Rightarrow \Pi$ unbounded

Clm: else, $P_1 = \{z : \forall i \quad z \le \beta_i/\alpha_i\}$ $\alpha_i > 0$

$= \{z : z \le \min_i \beta_i/\alpha_i\}$

$\Rightarrow$ Π feasible, bounded, $|\Pi| = \uparrow$

these are only options, hence equivalence. ◪

Cor: linear programming has finite time also

Sketch: apply Fourier-Motzkin elim to $z - \langle c, x\rangle \le 0$, $Ax \le b$ to elim x, compute / infeasible / unbounded.

Cor: Π $\max \langle c, x\rangle$ st $Ax \le b$, $x \ge 0$, $\amalg$ $\min \langle b, y\rangle$ st $y^T A \ge c$, $y \ge 0$

if Π feasible and bounded $\Rightarrow$ $\amalg$ feasible, bounded, and $|\Pi| = |\amalg|$

Pf. Π equiv to $\max \langle c, x\rangle$ st $A' x \le b'$

$$\begin{bmatrix} A \\ -I \end{bmatrix} \boxed{x} \le \begin{bmatrix} b \\ 0 \end{bmatrix}$$

Π feasible bounded $\rightarrow$ FM elim on $z - \langle c, x\rangle$, $A'x \le b'$ yields eqn $\alpha z \le \beta$ w/ – $\alpha > 0$, – $|\Pi| = \beta/\alpha$

– $(\alpha z \le \beta) = \alpha(z - \langle c, x\rangle) + (y')^T (A'x \le b')$, $y' \ge 0$

$\Rightarrow$ $(y')^T A' = \alpha c$

$(y')^T b' = \beta$

$\Rightarrow$ $(y'')^T A' = c$, $(y'')^T b' = |\Pi|$ $y'' = \frac{y'}{\alpha} \ge 0$ as $\alpha > 0$

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$$A' = \begin{bmatrix} A \\ \hline -I \end{bmatrix} \qquad b' = \begin{bmatrix} b \\ \hline 0 \end{bmatrix} \qquad y' = \begin{bmatrix} y \\ \hline w \end{bmatrix}$$

$\Rightarrow$ $c = (y')^T A' = y^T A - w^T \cdot I$ $\qquad y, w \geq 0$

$|\Pi| = y^T b + w^T \cdot 0$

$\Rightarrow$ $y^T A \geq c$, $\langle b, y \rangle = |\Pi|$ $\qquad y \geq 0$

$\Rightarrow$ $\amalg$ has feasible solution w/ value $|\Pi|$

$\Rightarrow$ $|\amalg| = |\Pi|$ [via weak duality] $\qquad \boxtimes$

Cor: max flow = mincut for arbitrary capacities

[they are dual LP values]

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next time - ideas for LP algos