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cs473

cs473 Algorithms: Lecture 19

logistics:
- pset 8 out tonight [due next Thursday]
- Midterm 2 Nov 11
- linear programs [generalizes max flow, min cut max flow]

last time:
- canonical form
- max flow, mincut

today:
- (weak) duality

ex: [last time]

[black tea] [jasmine green]

max $6x + 8y$

st.

$x \geq 0$ [nothing less]

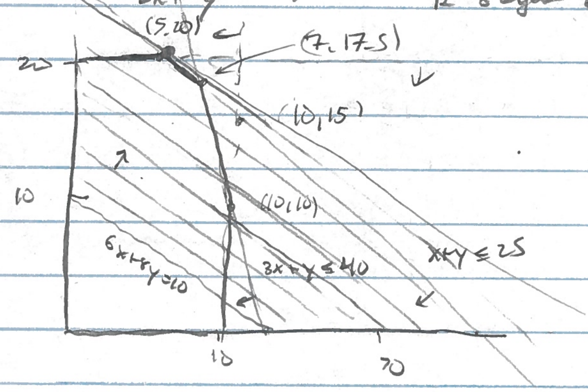
$y \geq 0$

$x \leq 10$ [# tea bags]

$y \leq 20$

$x + y \leq 25$ [bubbles]

$3x + y \leq 40$ [sugar]



$6 \cdot 5 + 8 \cdot 20 = 190$

$6 \cdot 7 + 8 \cdot 17.5 = 185$

$6 \cdot 10 + 8 \cdot 10 = 140$

Q: how to prove max $6x + 8y = 190$ formally?

idea: derive new equations from constraints

$x \leq 10$

$y \leq 20$

$x + y \leq 25$

$3x + y \leq 40$

$x, y \geq 0$

$$\begin{array}{r} x \leq 10 \\ +\ y \leq 20 \\ \hline x + y \leq 30 \end{array}$$

↳ vs $x + y \leq 25$

$$\begin{array}{r} 6x \leq 60 \\ 8y \leq 160 \\ \hline 6x + 8y \leq 220 \end{array}$$

objective function

$$\begin{array}{r} 8 \cdot (x + y) \leq 8 \cdot 25 \\ -2(x \geq 0) \\ \hline 6x + 8y \leq 200 \end{array}$$

objective

$8(x+y) \leq 8 \cdot 25 = 200$

$8x + 8y$ ≥] above objective
$6x + 8y$ as $x, y \geq 0$

$$\begin{array}{r} 2(3x + y) \leq 2 \cdot 40 \\ 6 \cdot y \leq 6 \cdot 20 \\ \hline 6x + 8y \leq 200 \end{array}$$

objective

$$\begin{array}{r} 6(x + y) \leq 6 \cdot 25 \\ 2y \leq 2 \cdot 20 \\ \hline 6x + 8y \leq 190 \end{array}$$

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so any feasible (x,y) has $6x+8y \le 190$
and: $(x,y)=(5,20)$ has $6x+8y=190$ ⟹ 190 is optimum value
feasible

Q: mechanize this process? to find best ub?

$a(\quad x \le 10 \quad) \longmapsto \quad ax \le 10a$ if $a \ge 0$
$b(\quad y \le 20 \quad)$ $b \ge 0$
$c(\quad x+y \le 25 \quad)$ $c \ge 0$
$d(\quad 3x+y \le 40 \quad)$ $d \ge 0$
$e(\quad x \ge 0 \quad)$ $e \le 0$
$f(\quad y \ge 0 \quad)$ $f \le 0$

$$\underbrace{(a+c+3d+e)}_{=6}x + \underbrace{(b+c+d+f)}_{=8}y \le \underbrace{10a+20b+25c+40d}_{\substack{\vee\!\vert \\ 6x+8y \text{ for } (x,y) \text{ feasible}}}$$

dual: min $10a+20b+25c+40d$
st $a,b,c,d \ge 0$
$e,f \le 0$ [not in canonical form]
$a+c+3d+e=6$
$b+c+d+f=8$

canonical form dual: min $10a+20b+25c+40d$
st $-(a+c+3d) \le -6$
$-(b+c+d) \le -8$
$a,b,c,d \ge 0$

any feasible (a,b,c,d) yields ub on $6x+8y$

hence:

$$\begin{array}{ll} \max & 6x+8y \\ \text{st} & x \le 10 \\ & y \le 20 \\ & x+y \le 25 \\ & 3x+y \le 40 \\ & x,y \ge 0 \end{array} \quad \le \quad \begin{array}{ll} \min & 10a+20b+25c+40d \\ \text{st} & -(a+c+3d) \le -6 \\ & -(b+c+d) \le -8 \\ & a,b,c,d \ge 0 \end{array}$$

is equality

$$6\cdot 5 + 8\cdot 20 = 190 = 10\cdot 0 + 20\cdot 2 + 25\cdot 6 + 40\cdot 0$$

and: # var : 2 = 4
nonaxis constraints = 4 = 2

def linear program (LP) Π = $\max \langle c,x\rangle$ st $Ax \le b$, $x \ge 0$ ← primal $c \in \mathbb{R}^n$ $A \in \mathbb{R}^{m\times n}$ $b \in \mathbb{R}^m$ ⟹ n vars
the dual $\amalg$

is min $\langle b,y\rangle$ ← m vars
st $A^T y \ge c$ ← n equation
$y \ge 0$

$(A^T)_{ij} = A_{ji}$
all vectors are column vectors

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rmk: - dual in canonical form

$$\begin{array}{c} \max \langle c, x\rangle \\ Ax \le b \\ x \ge 0 \\ A\ m\times n \end{array} \xrightarrow{\text{dual}} \begin{array}{c} -\max \langle -b, y\rangle \\ (-A^T)y \le -c \\ y \ge 0 \\ -A^T\ n\times m \end{array} \xrightarrow{\text{dual}} \boxed{\begin{array}{c} \max \langle c, z\rangle \\ Az \le b \\ z \ge 0 \\ A\ m\times n \end{array}} = \Pi$$

- many choices for "the" dual

ex:

$$\begin{array}{ll} \max & x \\ \text{st} & x \le 1 \\ & x \ge 0 \end{array} \mapsto \begin{array}{l} \min\ y \\ y \ge 1 \\ y \ge 0 \end{array}$$

$$\parallel$$

$$\begin{array}{ll} \max & x \\ \text{st} & x \le 1 \\ & x \le 2 \\ & x \ge 0 \end{array} \mapsto \begin{array}{l} \min\ y + 2z \\ y + z \ge 1 \\ y, z \ge 0 \end{array}$$

thm (weak duality): $|\Pi| \le |\amalg|$, if both feasible

$$\begin{array}{c} \max \langle c, x\rangle \\ Ax \le b \\ x \ge 0 \end{array} \qquad \begin{array}{c} \min \langle b, y\rangle \\ A^T y \ge c \\ y \ge 0 \end{array}$$

pf. x feasible for Π $\Rightarrow$ $Ax \le b$ $\Rightarrow$ $y^T A x \underset{y \ge 0}{\le} y^T b = \langle b, y\rangle$

y " $\amalg$ $\Rightarrow$ $y^T A \ge c^T$ $\qquad \underset{x\ge 0}{\ge} c^T x = \langle c, x\rangle$

hence $\langle c, x\rangle \le \langle b, y\rangle$ any x, y

$\Rightarrow |\Pi| = \max \langle c, x\rangle \le \min \langle b, y\rangle = |\amalg|$

rmk: dual unbounded $\Rightarrow$ primal infeasible

primal unbounded $\Rightarrow$ dual infeasible

if find x, y w/ $\langle c, x\rangle = \langle b, y\rangle$ then both are optimal!

thm (strong duality): if Π feasible and bounded then so is $\amalg$, and $|\Pi| = |\amalg|$

rmk: - $\amalg$ feasible $\Rightarrow$ Π feasible $\quad$ ∵ dual(dual) = primal ∵

- hence LPs always have short certificate of optimality

thm (weak duality): $\Pi = \begin{array}{c} \max \langle c, x\rangle \\ Ax = b \\ A'x \le b' \\ x \ge 0 \end{array} \le \begin{array}{c} \min \langle b, y\rangle + \langle b', z\rangle \\ A^T y + (A')^T z \ge c \\ z \ge 0 \end{array}$

∵ direct dual, now converting to canonical form ∵

pf. $x^T(A^T y + (A')^T z) \underset{x \ge 0}{\ge} x^T c = \langle c, x\rangle$

$= \underbrace{(xA)^T}_{=b} y + \underbrace{(xA')^T}_{\le (b')^T} z$

$\underset{z \ge 0}{\le} b^T y + (b')^T z$ $\quad$ ∵ no restrictions on y ∵

$= \langle b, y\rangle + \langle b', z\rangle$ $\qquad \boxtimes$

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$G=(V,E)$

prop: G network $s\neq t\in V$ $c: E\to \mathbb{R}_{\geq 0}$

max flow $= \max \sum_{s\to w} f_{sw} - \sum_{u\to s} f_{us}$

$$\text{s.t.} \quad \sum_{v\to w} f_{vw} - \sum_{u\to v} f_{uv} = 0 \quad \forall v\neq s,t$$

$$f_e \leq c(e)$$

$$f_e \geq 0$$

has dual $\min \sum c(e) x_e$

$$\text{s.t.} \quad d_v \leq d_u + x_{uv} \quad (u,v)\in E$$

$$d_s = 0$$

$$d_t = 1$$

$$x_{uv} \geq 0$$

Pf: first convert to $\max\ -F_t$ [[is equiv LP]]

s.t. unconstrained
$$\begin{cases} d_s\left(\sum_{s\to w} f_{sw} - \sum_{v\to s} f_{vs} - F_s = 0\right) \\ d_t\left(\sum_{t\to w} f_{tw} - \sum_{u\to t} f_{ut} - F_t = 0\right) \\ d_v\left(\sum_{v\to w} f_{vw} - \sum_{u\to v} f_{uv} = 0\right) \quad \forall v\neq s,t \end{cases}$$

$x_e \geq 0$ $\quad x_e\left(f_e \leq c(e)\right)$

$$f_e \geq 0$$

$$\min\ d_s\cdot 0 + d_t\cdot 0 + \sum_{v\neq s,t} d_v\cdot 0 + \sum_e x_e\cdot c(e)$$

s.t. F_s (unconstrained) $\quad d_s = 0$

F_t (unconstrained) $\quad d_t = 1$

$f_e = f_{uv}$ (≥ 0) $\quad d_u - d_v + x_{uv} \geq 0$

$$\equiv\ d_v \leq d_u + x_{uv}$$

$$x_e \geq 0$$

Prop: dual LP = min cut [[randomized rounding]]

Cor: strong duality $\Rightarrow$ max flow = min cut [[LPs always feasible]]

rmk: $\hookrightarrow$ this works for arbitrary capacities

Ford Fulkerson required integer/rational capacities to terminate $\leftarrow$ used to prove maxflow = mincut

logistics: - pset 8 out tonight
- midterm 2 Nov 11

today: - dual LP
- weak duality
- strong duality $\Rightarrow$ max flow = min cut

next time: strong duality