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2019-10-24.4 → 2019-10-29.7
2019-10-29.2 ←
CS473

CS473 Algorithms: Lecture 18

logistics:
- pset 7 due W10
- midterm 2 Nov 11

last time:
- flow decomposition
- edge disjoint paths
- baseball elimination
- min cost max flow

today:
- linear programming

min cost max flow

$\leq_L$ $\leq_L$ $\geq$

max-flow $\quad$ shortest paths $\quad$ min cost bipartite perfect matching

$\leq_L$

max bipartite matching

Q: generalize further? [one algo to rule them all?] [specialized algo can still be better]

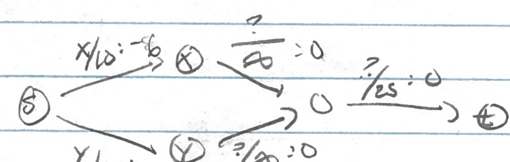
ex: opening new bubble tea shop [how to maximize profit]

two offerings
- black tea w/ tapioca pearls ← # gallons $= x \geq 0$
- jasmine green tea w/ tapioca pearls ← # liters $= y \geq 0$

constraints
- $x \leq 10$ [# green tea bags]
- $y \leq 20$ [# black tea bags]
- $x + y \leq 25$ [≤ 30] [amount of pearls] [$x + y \leq 30$ clearly]
- $3x + y \leq 40$ [amount of sugar]

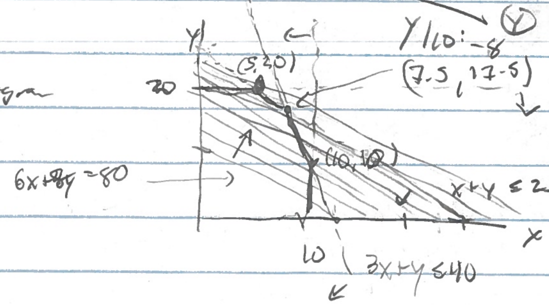
maximize $\underbrace{6}_{\text{price per gallon}}x + 8y$

min cost max flow



[w/o sugar]

linear program



$6 \cdot 5 + 8 \cdot 20 = 190$

$6 \cdot 7.5 + 8 \cdot 17.5 = 185$

$6 \cdot 10 + 8 \cdot 10 = 140$

rmk:
- continuous region to optimize over
- optimizing linear function
- optimum is at extreme points

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def A linear program is given by $c \in \mathbb{R}^n$, $A_\le, A_=, A_\ge \in \mathbb{R}^{m \times n}$, $b_\le, b_=, b_\ge \in \mathbb{R}^m$
and asks for

$$\Pi \begin{cases} \max \sum_{i=1} c_i x_i & \leftarrow \langle c, x\rangle \\ \text{st} & \\ \forall 1 \le i \le m \; \sum_{j=1}^n (A_\le)_{ij} x_j \le (b_\le)_i & \leftarrow A_\le \cdot x \le b_\le \\ \forall 1 \le i \le m \; \sum_{j=1}^n (A_=)_{ij} x_j = (b_=)_i & A_= \cdot x = b_= \\ \forall 1 \le i \le m \; \sum_{j=1}^n (A_\ge)_{ij} x_j \ge (b_\ge)_i & A_\ge \cdot x \ge b_\ge \end{cases}$$

coordwise

input size: n = # variables
m = # constraints [$3m$ in above]
bit complexity of $c, A_\le, A_=, A_\ge, b_\le, b_=, b_\ge$

Π is feasible if exists $x \in \mathbb{R}^n$ satisfying constraints $\leftarrow$ feasible point
else infeasible

Π is bounded if $|\Pi| < \infty$, else unbounded.

Q: given Π compute $|\Pi|$?

rmk: - linear "programming" [like dynamic "programming"]
- could also minimize $\min \langle c, x\rangle = -\max \langle -c, x\rangle$
- m not bounded by function of n [$m \le \text{poly}(n)$ for graph problems]

ex: in $\mathbb{R}^2$

$\leftarrow$ circle needs ∞ many constraints

- continuous optimization $\Rightarrow$ no apriori obvious brute force algo

def. A canonical form linear program is of the form

$$\max \langle c, x\rangle \quad \text{st} \quad Ax \le b, \; x \ge 0$$

lem: Π linear program $\max \langle c, x\rangle$
$A_\le x \le b_\le$
$A_= x = b_=$
$A_\ge x \ge b_\ge$

[like reductions for max flow]

then exists canonical Π' $\max \langle c', x'\rangle$ st $A'x' \le b'$, $x' \ge 0$

st: x feasible in Π $\mapsto$ x' feasible in Π' - $\langle c, x\rangle = \langle c', x'\rangle$
- x' computable efficiently
" $\leftarrow$ " "

so $|\Pi| = |\Pi'|$

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Pf: $A_{\geq} x \geq b_{\geq} \equiv (-A_{\geq}) x \leq (-b_{\geq})$

$A_{=} x = b_{=} \equiv \begin{bmatrix} A_{=} \\ -A_{=} \end{bmatrix} x \leq \begin{bmatrix} b_{=} \\ -b_{=} \end{bmatrix}$ ⟦ block matrix ⟧

hence may assum Π is $\max \langle c, x \rangle$
st $Ax \leq b$

create x' as $\{x_1^-, x_1^+, x_2^-, x_2^+, \ldots, x_n^-, x_n^+\}$ ⟦ $2n$ variables ⟧

$x_i^- = \begin{cases} -x_i & x_i < 0 \\ 0 & x_i \geq 0 \end{cases}$ $\quad x_i^+ = \begin{cases} 0 & x_i < 0 \\ x_i & x_i \geq 0 \end{cases}$

$\Rightarrow x_i = x_i^+ - x_i^- \quad x_i^+, x_i^- \geq 0$

$A' := A(x^+ - x^-) \leq b$

c': $\langle c', x' \rangle = \langle c, x^+ - x^- \rangle = \langle c, x^+ \rangle - \langle c, x^- \rangle$

x feasible $\equiv Ax \leq b \Rightarrow x = x^+ - x^-$, $x^+, x^- \geq 0$ $\quad A(x^+ - x^-) \leq b$, $x^+, x^- \geq 0$, $\langle c', x' \rangle = \langle c, x \rangle$ ⟦ and efficient ⟧

x' feasible $\equiv A(x^+ - x^-) \leq b$, $x^+, x^- \geq 0 \mapsto x = x^+ - x^-$ $\quad Ax \leq b$, $\langle c, x \rangle = \langle c', x' \rangle$ ∎

Prop: network directed $G = (V, E)$ $s \neq t \in V$ $c: E \to \mathbb{R}_{\geq 0}$

max flow $= \max \sum_{s \to w} f_{sw} - \sum_{u \to s} f_{us}$

st $\forall e \in E \quad f_e \geq 0$ ⟦ capacity ⟧

$\# \text{vars} = |E|$ $\quad f_e \leq c_e$

$\# \text{constraints} = 2 \cdot |E| + (|V| - 2)$ $\quad \forall v \in V \setminus \{s,t\} \underbrace{\sum_{v \to w} f_{vw}}_{f^{out}(v)} - \underbrace{\sum_{u \to v} f_{uv}}_{f^{in}(v)} \geq 0$ ⟦ conservation

∎

Prop: min cost max flow $\quad \max - \sum_e f_e \cdot p_e$

st: $\sum_{s \to w} f_{sw} - \sum_{u \to s} f_{us} = |f^*|$ max flow value

...

Prop: network $G = (V, E)$ $s \neq t \in V$ $c: E \to \mathbb{R}_{\geq 0}$

$\min_{\substack{V = S \sqcup T \\ s \in S, \, t \in T}} c(S, T) = \min \sum_{u \to v} c(u \to v) x_{u,v}$

$c(S,T) = \sum_{u \in S, v \in T} c(u \to v)$

st: $\forall e \in E \quad x_e \geq 0$

$\forall v \in V$

$d_s = 0$

$d_t = 1$

$\forall (u,v) \in E \quad d_v \leq d_u + x_e$

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Pf: $\geq$: given $V = \underset{\ni s}{S} \sqcup \underset{\ni t}{T}$ define $d(v) = \begin{cases} 0 & v \in S \\ 1 & v \in T \end{cases}$

$\Rightarrow d(s) = 0$
$d(t) = 1$

$$x_e = \left[\begin{cases} 1 & e = u \to v \quad u \in S, v \in T \\ 0 & \text{else} \end{cases}\right]$$

Clm: $\forall e = (u \to v) \quad d(v) \leq d(u) + x_e$

Pf:

$d(v)$	$d(u)$	x_e
0	0	0
1	1	0
1	0	0
0	1	1

$$c(S,T) = \sum_{u \in S, v \in T} c(u \to v) = \sum_{u,v} c(u \to v) x_e \geq \min \text{ "}$$

$\leq$: given d w/ $d(s) = 0$ $d(t) = 1$ $d(v) \leq d(u) + x_{uv}$ $(u,v) \in E$

possibly fractional

idea: randomized rounding

algo: pick $\theta \in (0,1]$ uniformly
$S = \{v : d(v) < \theta\}$
output $V = S^{\ni s} \sqcup T^{\ni t}$

Clm: $\mathbb{E}[c(S,T)] \leq |\Pi|$

$\Rightarrow$ exists $V = S \sqcup T$ w/ $c(S,T) \leq |\Pi|$ [probabilistic method]

Pf: $\mathbb{E}[c(S,T)] = \sum_{u,v} \mathbb{E}[c(u \to v) \cdot \mathbb{1}[u \in S, v \in T]]$
$= c(u \to v) \cdot \Pr[u \in S, v \in T]$

$u \in S, v \in T \Rightarrow d(u) < \theta \leq d(v) \leq d(u) + x_{uv}$ note: $d(v) < d(u)$ is possible $\Rightarrow u \to v$ not cut ever

$\Rightarrow \theta \in (d(u), d(u) + \underbrace{x_{uv}}_{\geq 0}]$

as $x_{uv} \geq 1$ is possible, as well as $d(v) < d(u)$ and also $d(v), d(u) \notin [0,1]$

hence $\Pr[u \in S, v \in T] \leq x_{uv}$

$$\Rightarrow \mathbb{E}[c(S,T)] \leq \sum_{(u,v) \in E} c(u \to v) x_{uv} = |\Pi| \quad \blacksquare$$

rmk: - not randomized algo as did not give probability of success (was existential)
- does give efficient deterministic algo: try all $\theta \in (0,1] \cap \{d(v)\}_{v \in V}$ [these are the only ones that matter]

Logistics - pset 2 due 11/10
- midterm 2 Nov 11

today: - linear programs
- canonical form
- max flow, min cut

next time: weak duality