CS 473: Algorithms, Fall 2018

Entropy, Randomness, and Information

Lecture 26

December 3, 2018

26.1: Entropy

Quote

"If only once - only once - no matter where, no matter before what audience - I could better the record of the great Rastelli and juggle with thirteen balls, instead of my usual twelve, I would feel that I had truly accomplished something for my country. But I am not getting any younger, and although I am still at the peak of my powers there are moments - why deny it? - when I begin to doubt - and there is a time limit on all of us."
–Romain Gary, The talent scout.

Entropy: Definition

Definition

The ***entropy*** in bits of a discrete random variable $\boldsymbol{X}$ is

$$\mathbb{H}(\boldsymbol{X}) = -\sum_{\boldsymbol{x}} \mathbf{Pr}\Big[\boldsymbol{X} = \boldsymbol{x}\Big] \lg \mathbf{Pr}\Big[\boldsymbol{X} = \boldsymbol{x}\Big].$$

Equivalently, $\mathbb{H}(\boldsymbol{X}) = \mathbf{E}\Big[\lg \frac{1}{\mathbf{Pr}[\boldsymbol{X}]}\Big]$.

Entropy

Clicker question

Consider X a random variable that picks its value uniformly from $1, \ldots, n$. We have that its entropy $\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$ is

1. $O(\log n)$.
2. $O(n)$.
3. $\ln n$.
4. $n * \ln n$.
5. $\lg n$.

Entropy intuition...

Intuition...

$\mathbb{H}(X)$ is the number of ***fair*** coin flips that one gets when getting the value of X.

Interpretation from last lecture...

Consider a (huge) string $S = s_1 s_2 \ldots s_n$ formed by picking characters independently according to X. Then

$$|S|\,\mathbb{H}(X) = n\mathbb{H}(X)$$

is the minimum number of bits one needs to store the string S (when we compress it).

Entropy II

Clicker question

Consider X a random variable that

$$\Pr[X = i] = \frac{1/i}{\alpha},$$

for $i = 1, \ldots, \infty$, where $\alpha = \sum_{i=1}^{\infty} 1/i$.
The entropy of X is
$\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$ equal to

1. $O(1)$.
2. $O(n)$.
3. 0.
4. ∞.

Entropy IV

Clicker question

Consider X a random variable that

$$\Pr[X = i] = \frac{1/i^2}{\alpha},$$

for $i = 2, \ldots, \infty$, where $\alpha = \sum_{i=2}^{\infty} 1/i^2$.
The entropy of X is
$\mathbb{H}(X) = -\sum_x \Pr\Big[X = x\Big] \lg \Pr\Big[X = x\Big]$ equal to

1. $O(1)$.
2. $O(n)$.
3. 0.
4. ∞.

Entropy V

Clicker question

Consider X a random variable that

$$\Pr[X = i] = 2^{-i}$$

for $i = 1, \ldots, \infty$. The entropy of X is $\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$ equal to

1. $O(1)$.
2. $O(n)$.
3. 0.
4. ∞.
5. $\lg n$.

Entropy of a geometric distribution...

$$\mathbb{H}(X) = -\sum_{x} \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$$

$$= -\sum_{i=1}^{\infty} \frac{1}{2^i} \lg \frac{1}{2^i}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \lg 2^i$$

$$= \sum_{i=1}^{\infty} \frac{i}{2^i}$$

$$= 2.$$

Entropy of a geometric distribution...

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\mathbb{H}(\boldsymbol{X}) &= -\sum_{\boldsymbol{x}} \mathbf{Pr}\Big[\boldsymbol{X} = \boldsymbol{x}\Big] \, \mathbf{lg}\, \mathbf{Pr}\Big[\boldsymbol{X} = \boldsymbol{x}\Big] \\
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&= \sum_{i=1}^{\infty} \frac{\mathbf{1}}{\mathbf{2}^i} \, \mathbf{lg}\, \mathbf{2}^i \\
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&= \mathbf{2}.
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Binary entropy

$$\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$$

$\Longrightarrow$

Definition

The ***binary entropy*** function $\mathbb{H}(p)$ for a random binary variable that is 1 with probability p, is $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$. We define $\mathbb{H}(0) = \mathbb{H}(1) = 0$.

Q: How many truly random bits are there when given the result of flipping a single coin with probability p for heads?

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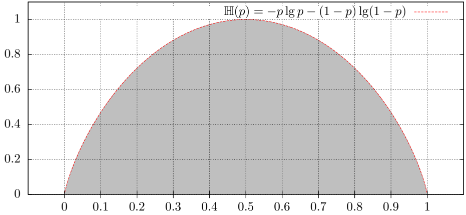
Binary entropy: $\mathbb{H}(\mathbf{p}) = -\mathbf{p}\lg\mathbf{p} - (1-\mathbf{p})\lg(1-\mathbf{p})$



1. $\mathbb{H}(p)$ is a concave symmetric around $1/2$ on the interval $[0,1]$.
2. maximum at $1/2$.
3. $\mathbb{H}(3/4) \approx 0.8113$ and $\mathbb{H}(7/8) \approx 0.5436$.
4. $\Longrightarrow$ coin that has $3/4$ probably to be heads have higher amount of “randomness” in it than a coin

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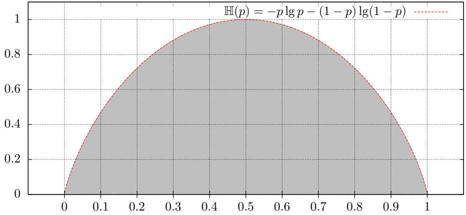

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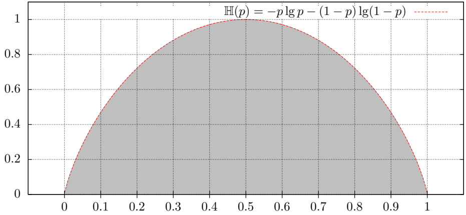

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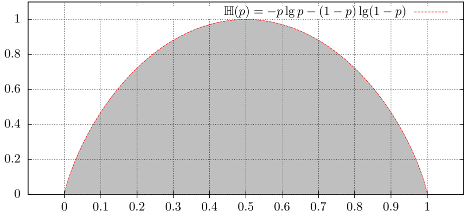

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And now for some unnecessary math

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2. $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$
3. $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
4. $\implies \mathbb{H}''(p) \leq 0$, for all $p \in (0,1)$, and the $\mathbb{H}(\cdot)$ is concave.
5. $\mathbb{H}'(1/2) = 0 \implies \mathbb{H}(1/2) = 1 \max$ of binary entropy.
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26.3: Squeezing randomness

Task at hand: Squeezing good random bits...

...out of bad random bits...

1. $b_1, \ldots, b_n$: result of n coin flips...
2. From a faulty coin!
3. p: probability for head.
4. We need fair bit coins!
5. Convert $b_1, \ldots, b_n \implies b'_1, \ldots, b'_m$.
6. **New bits must be truly random**: Probability for head is $1/2$.
7. **Q:** How many truly random bits can we extract?

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Intuitively...

Squeezing good random bits out of bad random bits...

Question...

Given the result of n coin flips: $b_1, \ldots, b_n$ from a faulty coin, with head with probability p, how many truly random bits can we extract?

If believe intuition about entropy, then this number should be $\approx n\mathbb{H}(p)$.

Back to Entropy

1. ***entropy*** of X is
   $\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$.
2. Entropy of uniform variable..

Example

A random variable X that has probability $1/n$ to be i, for $i = 1, \ldots, n$, has entropy
$\mathbb{H}(X) = -\sum_{i=1}^{n} \frac{1}{n} \lg \frac{1}{n} = \lg n$.

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Flipper

Clicker question

You are given a coin that is head with probability p, and tail with probability $q = 1 - p$. We flip it three times, and get the string $S = s_1 s_2 s_3$. We have the following:

1. $\mathbf{Pr}[S = 001] = \mathbf{Pr}[S = 011] = pq^2$.
2. $\mathbf{Pr}[S = 101] = \mathbf{Pr}[S = 110] = \mathbf{Pr}[S = 011] = pq^2$.
3. $\mathbf{Pr}[S = 111] = \mathbf{Pr}[S = 000] = q^3$.
4. $\mathbf{Pr}[S = 001] = \mathbf{Pr}[S = 010] = \mathbf{Pr}[S = 100] = pq^2$.
5. $\mathbf{Pr}[S = 000] + \mathbf{Pr}[S = 111] = (p + q)^3$.

Lemma: Entropy additive for independent variables

Lemma

Let X and Y be two independent random variables, and let Z be the random variable (X, Y). Then $\mathbb{H}(Z) = \mathbb{H}(X) + \mathbb{H}(Y)$.

Proof

In the following, summation are over all possible values that the variables can have. By the independence of $\boldsymbol{X}$ and $\boldsymbol{Y}$ we have

$$
\begin{aligned}
\mathbb{H}(\boldsymbol{Z}) &= \sum_{x,y} \Pr\Big[(\boldsymbol{X}, \boldsymbol{Y}) = (x, y)\Big] \lg \frac{1}{\Pr[(\boldsymbol{X}, \boldsymbol{Y}) = (x, y)]} \\
&= \sum_{x,y} \Pr\Big[\boldsymbol{X} = x\Big] \Pr\Big[\boldsymbol{Y} = y\Big] \lg \frac{1}{\Pr[\boldsymbol{X} = x] \Pr[\boldsymbol{Y} =
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{x} \sum_{y} \Pr[\boldsymbol{X} = x] \Pr[\boldsymbol{Y} = y] \lg \frac{1}{\Pr[\boldsymbol{X} = x]} \\
&\quad + \sum_{y} \sum_{x} \Pr[\boldsymbol{X} = x] \Pr[\boldsymbol{Y} = y] \lg \frac{1}{\Pr[\boldsymbol{Y} = y]}
\end{aligned}
$$

Proof continued

$$
\begin{aligned}
\mathbb{H}(Z) &= \sum_x \sum_y \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[X = x]} \\
&\qquad + \sum_y \sum_x \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[Y = y]} \\
&= \sum_x \Pr[X = x] \lg \frac{1}{\Pr[X = x]} \\
&\qquad + \sum_y \Pr[Y = y] \lg \frac{1}{\Pr[Y = y]} \\
&= \mathbb{H}(X) + \mathbb{H}(Y).
\end{aligned}
$$

The entropy of Y...

Clicker question

Consider a binary string Y generated by flipping a coin n times, where the probability for heads is p. Then we have that

1. $\mathbb{H}(Y) = \ln \binom{n}{np}$.
2. $\mathbb{H}(Y) = np$.
3. $\mathbb{H}(Y) = n\mathbb{H}(p)$.
4. $\mathbb{H}(Y) = n - n\mathbb{H}(p)$.
5. $\mathbb{H}(Y) = \mathbb{H}(np)$.

Bounding the binomial coefficient using entropy

Lemma

$q \in [0, 1]$

nq is integer in the range $[0, n]$.

Then

$$\frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{nq} \leq 2^{n\mathbb{H}(q)}.$$

Proof

Holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We have

$$\binom{n}{nq} q^{nq}(1-q)^{n-nq} \le (q + (1-q))^n = 1.$$

We also have: $q^{-nq}(1-q)^{-(1-q)n} = 2^{n\,(-q\lg q-(1-q)\lg(1-q))} = 2^{n\mathbb{H}(q)}$, we have

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$$\binom{n}{nq} \leq q^{-nq}(1-q)^{-(1-q)n} = 2^{n\mathbb{H}(q)}.$$

Proof

Holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We have

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Flipper revisited...

Clicker question

p: coin returns head with this probability. $q = 1 - p$. Flip coin n times, let X be the resulting string. Assume np and nq are integer.
$\mathcal{S}_i$: set of all binary strings length n with i ones in them. Then:

1. $\mathbf{Pr}[X \in \mathcal{S}_i]$ is maximal for $i = np$.
2. $\forall s, s' \in S_i$, we have $\mathbf{Pr}[X = s] = \mathbf{Pr}[X = s'] = \binom{n}{i} p^i q^{n-i}$.
3. If $X \in \mathcal{S}_i$ then entropy of X is $\lg \binom{n}{i}$.
4. $\mathbb{H}(X) = n\mathbb{H}(p)$
5. All of the above.

Generalization...

Corollary

We have:

1. $q \in [0, 1/2] \Rightarrow \binom{n}{\lfloor nq \rfloor} \leq 2^{n\mathbb{H}(q)}$.
2. $q \in [1/2, 1]$ $\binom{n}{\lceil nq \rceil} \leq 2^{n\mathbb{H}(q)}$.
3. $q \in [1/2, 1] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lfloor nq \rfloor}$.
4. $q \in [0, 1/2] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lceil nq \rceil}$.

Proof is straightforward but tedious.

What we have...

1. Proved that $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$.
2. Estimate is loose.
3. Sanity check...
   3.1 A sequence of n bits generated by coin with probability q for head.
   3.2 By Chernoff inequality... roughly nq heads in this sequence.
   3.3 Generated sequence Y belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences .
   3.4 ...of similar probability.
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Just one bit...

question

Given a coin C with:
p: Probability for head.
$q = 1 - p$: Probability for tail.
Q: How to get <u>**one**</u> true random bit, by flipping C.

Describe an algorithm!

Extracting randomness...

Entropy can be interpreted as the amount of unbiased random coin flips can be extracted from a random variable.

Definition

An extraction function **Ext** takes as input the value of a random variable X and outputs a sequence of bits y, such that $\mathbf{Pr}\Big[\mathsf{Ext}(X) = y \;\Big|\; |y| = k\Big] = \frac{1}{2^k}$, whenever $\mathbf{Pr}[|y| = k] > 0$, where $|y|$ denotes the length of y.

Extracting randomness...

1. X: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of x.
3. Def. subtle: all extracted seqs of same len have same probability.
4. Another example of extraction scheme:
   4.1 X: uniform random integer variable $0, \ldots, 11$.
   4.2 $\mathsf{Ext}(x)$: output the binary representation for x if $0 \le x \le 7$.
   4.3 If x is between 8 and 11?
   4.4 Idea... Output binary representation of $x - 8$ as a two bit number.
5. A valid extractor...

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Technical lemma

The following is obvious, but we provide a proof anyway.

Lemma

Let x/y be a faction, such that $x/y < 1$. Then, for any i, we have $x/y < (x+i)/(y+i)$.

Proof.

We need to prove that $x(y+i) - (x+i)y < 0$. The left size is equal to $i(x-y)$, but since $y > x$ (as $x/y < 1$), this quantity is negative, as required. $\square$

A uniform variable extractor...

Theorem

1. X: *random variable chosen uniformly at random from* $\{0, \ldots, m-1\}$.
2. *Then there is an extraction function for* X:
   2.1 *outputs on average at least*

$$\lfloor \lg m \rfloor - 1 = \lfloor \mathbb{H}(X) \rfloor - 1$$

independent and unbiased bits.

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Proof

1. m: A sum of unique powers of 2, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:
3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of 2.
4. If x is in block 2^k, output its relative location in the block in binary representation.
5. Example: $x = 10$:
   then falls into block 2^2...
   x relative location is 2. Output 2 written using two bits,
   Output: "10".

Proof

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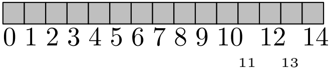
2. Example: 

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   x relative location is 2. Output 2 written using two bits

Proof

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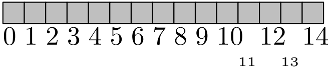
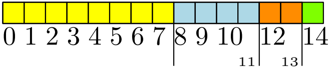
2. Example: 



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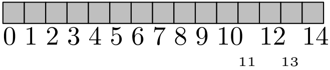
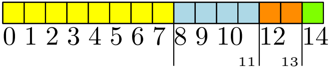
2. Example: 



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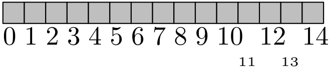

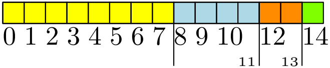

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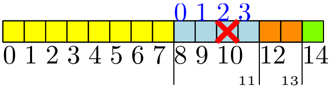

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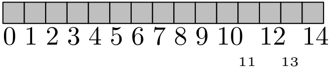
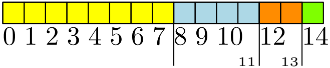
2. Example: 



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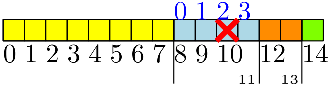

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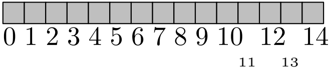
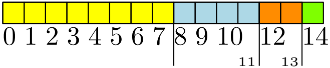
2. Example: 



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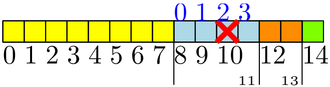
5. Example: $x = 10$: 

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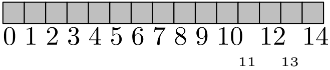
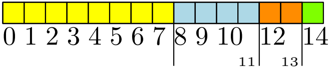
2. Example: 



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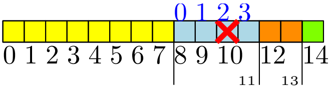

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2. Theorem holds if m is a power of two. Only one block.
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   ... entropy is k.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \lfloor \lg(m - 2^k) \rfloor < k$.

   There must be a block of size u in the decomposition of m.
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1. By lemma, since $\frac{m-2^k}{m} < 1$:

$$\frac{m-2^k}{m} \leq \frac{m-2^k+\left(2^{u+1}+2^k-m\right)}{m + \left(2^{u+1}+2^k-m\right)} = \frac{2^{u+1}}{2^{u+1}+2^k}.$$

2. By induction (assumed holds for all numbers smaller than m):

$$\mathbf{E}[Y] \geq \frac{2^k}{m}k + \frac{m-2^k}{m}\left(\underbrace{\lfloor \lg(m-2^k)\rfloor}_{u} - 1\right)$$

$$= \frac{2^k}{m}k + \frac{m-2^k}{m}(\underbrace{k-k}_{=0} + u - 1)$$

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1. We have:

$$\begin{aligned} \mathbf{E}\Big[Y\Big] &\geq k + \frac{m-2^k}{m}(u-k-1) \\ &\geq k + \frac{2^{u+1}}{2^{u+1}+2^k}(u-k-1) \\ &= k - \frac{2^{u+1}}{2^{u+1}+2^k}(1+k-u), \end{aligned}$$

   since $u-k-1 \leq 0$ as $k > u$.

2. If $u = k-1$, then $\mathbf{E}[Y] \geq k - \frac{1}{2} \cdot 2 = k-1$, as required.
3. If $u = k-2$ then $\mathbf{E}[Y] \geq k - \frac{1}{3} \cdot 3 = k-1$

Proof continued..

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$$\mathbf{E}\Big[\boldsymbol{Y}\Big] \geq \boldsymbol{k} + \frac{\boldsymbol{m} - \mathbf{2}^{\boldsymbol{k}}}{\boldsymbol{m}}(\boldsymbol{u} - \boldsymbol{k} - \mathbf{1})$$
$$\geq \boldsymbol{k} + \frac{\mathbf{2}^{\boldsymbol{u}+\mathbf{1}}}{\mathbf{2}^{\boldsymbol{u}+\mathbf{1}} + \mathbf{2}^{\boldsymbol{k}}}(\boldsymbol{u} - \boldsymbol{k} - \mathbf{1})$$
$$= \boldsymbol{k} - \frac{\mathbf{2}^{\boldsymbol{u}+\mathbf{1}}}{\mathbf{2}^{\boldsymbol{u}+\mathbf{1}} + \mathbf{2}^{\boldsymbol{k}}}(\mathbf{1} + \boldsymbol{k} - \boldsymbol{u}),$$

   since $\boldsymbol{u} - \boldsymbol{k} - \mathbf{1} \leq \mathbf{0}$ as $\boldsymbol{k} > \boldsymbol{u}$.

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$$\begin{aligned}
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&= k - \frac{k-u+1}{2^{k-u-1}} \\
&= k - \frac{2+(k-u-1)}{2^{k-u-1}} \\
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   since $(2+i)/2^i < 1$ for $i > 2$.

Proof continued.....

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