11/27/18

Approximation algorithms using LP

weighted Vertex Cover –

$G=(V,E)$

$\forall v\in V \quad w(v)\geq 0$

Find minimum weight vertex cover.

NP-Hard

Integer program

$v\in V \quad x_v\in\{0,1\}$
$\forall e=uv\in E \quad x_u+x_v\geq 1$
$\min \sum_v w(v)x_v$

Relaxion

Linear program

$\min \sum_v w(v)x_v$

$\forall uv\in E \quad x_u+x_v\geq 1$ ←

$\forall v\in V \quad 0\leq x_v \leq 1$ (redundant)

$\hat{x}_v \equiv$ Value of x_v in the LP solution

$\hat{\alpha} = \sum w(v)\hat{x}_v$ Value of the LP

$\alpha^I \equiv$ Value of optimal integral solution

$$\hat{\alpha}\leq\alpha^I$$

$\hat{x}_v=0.4$

$\hat{x}_v=1$

$uv\in E \quad \hat{x}_u+\hat{x}_v\geq 1$

$\Rightarrow \hat{x}_u\geq\frac{1}{2}$ or $\hat{x}_v\geq\frac{1}{2}$

Multiply by 2

$\bar{x}_u=\min(1, 2\hat{x}_u) \quad \forall u\in V$

$S=\{v\in V \mid \bar{x}_v=1\}$

Claim S is a vertex cover.

Claim $\sum_{s\in S} w(s)\leq\sum_{v\in V}\bar{x}_v w(v)$

$\leq\sum_{v\in V} 2\hat{x}_v w(v)=2\cdot\hat{\alpha}$ ← LP value.

$\hat{\alpha}=\sum_{v\in V} w(v)\hat{x}_v$

$\to 2\sum_{v\in V}\hat{x}_v w(v)=2\hat{\alpha}\leq 2\alpha^I$.

$\hat{\alpha}\leq\alpha^I$

$\alpha\leq\alpha^I\leq$ computed sol $\leq 2\alpha\leq 2\alpha^I$

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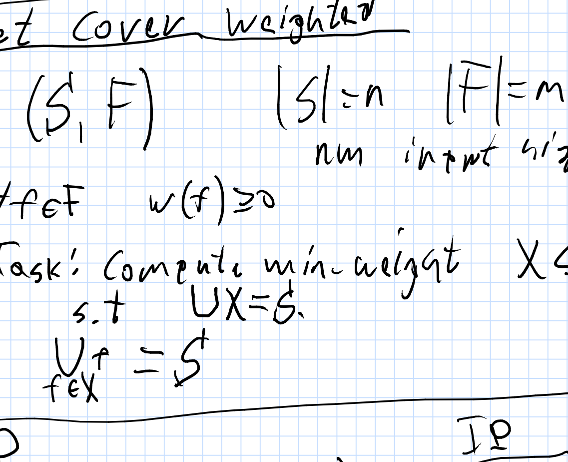
very expensive



Set Cover weighted

$(S, F) \quad |S|=n \quad |F|=m$

nm input size

$\forall f\in F \quad w(f)\geq 0$

Task: Compute min-weight $X\subseteq F$ s.t. $\cup X=S$.

$\bigcup_{f\in X} f=S$

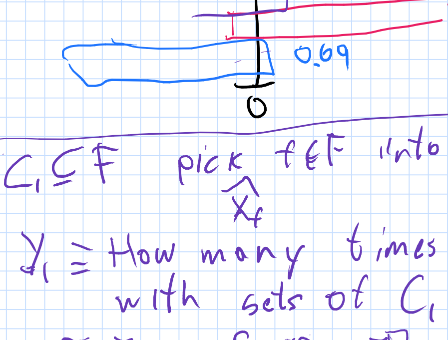
LP

$\min \sum_{f\in F} x_f w(f)$

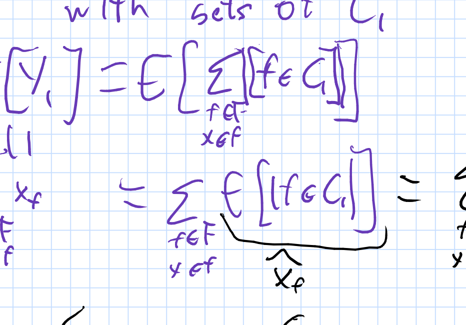
$\forall s\in S \quad \sum_{f\in F, s\in f} x_f\geq 1$

$\forall f\in F \quad x_f\geq 0$

IP: $x_f\in\{0,1\}$



Quite possible $\forall f\in F \quad \hat{x}_f\leq\frac{1}{\log^2 n}$



$C_1\subseteq F$ pick $f\in F$ into C_1 with prob $\hat{x}_f$

$Y_1\equiv$ How many times x is covered with sets of C_1

$E[Y_1]=E\left[\sum_{f\in F, x\in f}[f\in C_1]\right]$

$\sum_{f\in F, x\in f} x_f = \sum_{f\in F, x\in f} E[[f\in C_1]] = \sum_{f\in F, x\in f}\hat{x}_f\geq 1$ (ineq in the LP)

$C_2, C_3, \ldots, C_m \quad m=O(\log n)$

independent samples

Claim $P\left[x \text{ is not covered by } C_1\right]\leq\frac{1}{e}$

Proof

$P[x \text{ not covered}]=P\left[\text{all sets covering } x \text{ not picked}\right]$

$=\prod_{f\in F, x\in f} P[f \text{ is not chosen}]$ $\quad 1-x\leq e^{-x}$

$=\prod_{f\in F, x\in f}(1-\hat{x}_f)\leq\prod\exp(-\hat{x}_f)$

$=\exp\left(-\sum\hat{x}_f\right)$

$\leq\exp(-1)=\frac{1}{e}\leq\frac{1}{2}$.

$P\left[x \text{ is not covered by } C_1,\ldots,C_m\right]$

$\leq\prod_{i=1}^{m} P\left[x \text{ is not covered by } C_i\right]$

$\leq\frac{1}{2^m}<\frac{1}{n^{10}}$

$P[\text{failure}]=P[\exists \text{ element that is not covered}]$

union bound

$\leq|S|\,P[\text{specific element is not covered}]$

$\leq n\cdot\frac{1}{n^{10}}=\frac{1}{n^9}$.

Price of solution

$E[w(C_1)]=E\left[\sum_{f\in F}[f\in C_1]w(f)\right]$

$=\sum_{f\in F} E[[f\in C_1]]\,w(f)$ (where $E[[f\in C_1]]=\hat{x}_f$)

$=\sum_{f\in F}\hat{x}_f w(f)=\hat{\alpha}$ ← LP value

$E[w(C_1\cup\cdots\cup C_m)]=m\hat{\alpha}=O(\hat{\alpha}\log n)$

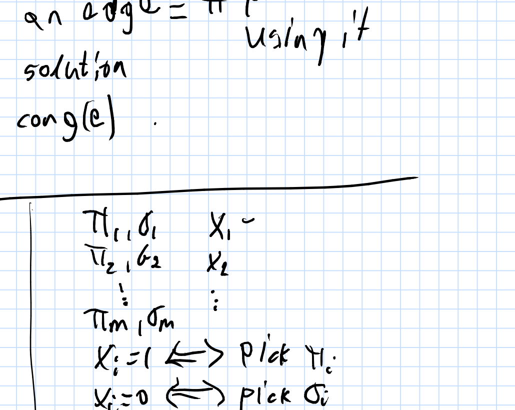
$=O(\alpha^I\log n)$ $\quad 20\hat{\alpha}\log n \quad 40\hat{\alpha}\log n$

Pick f into cover with probability

$\min(\hat{x}_f\cdot m, 1)$

$x \quad \sum_{f\in F, x\in f} m\hat{x}_f\geq m$

Congestion



congestion of an edge $\equiv$ # paths selected using it

Congestion of solution

$=\max_{e\in E}\text{cong}(e)$

$\min g$

$\forall e\in E$

$\pi_1, \sigma_1 \quad x_1$

$\pi_2, \sigma_2 \quad x_2$

$\vdots$

π_m, σ_m

$x_i=1 \Leftrightarrow$ pick π_i

$x_i=0 \Leftrightarrow$ pick σ_i

$\sum_{e\in\pi_i} x_i+\sum_{e\in\sigma_i}(1-x_i)\leq g \quad \forall e\in E$
$x_i\geq 0, g\geq 0$
$\min g$

$\hat{x}_i$ LP value

Pick π_i with prob. $\hat{x}_i \quad \forall i$

otherwise pick σ_i

$\forall e \quad \sum_{\pi_i}\hat{x}_i+\sum_{\sigma_i, e\in\sigma_i}(1-\hat{x}_i)\leq\hat{g}$

$\hat{g}\geq 10\log n$