Linear Programming II

min x^*

$x^* + \sum \alpha_i x_i \geq c$

Lecture 19

October 29, 2018

Ex.

$\sum x_i \geq c_i$

LP feasibility...

Clicker question

Let $\mathcal{L}$ be an instance of LP with n variables and m constraints. Then we have the following:

1. $\mathcal{L}$ is always feasible.
2. $\mathcal{L}$ might not be feasible, but it can be made feasible by changing the value of one of the variables.
3. $\mathcal{L}$ might not be feasible, but can be fixed by adding a single variable with the appropriate value.
4. $\mathcal{L}$ might not be feasible, but can be fixed by adding two variable with the correct value (one need two variables because of the equality constraints).
5. $\mathcal{L}$ might not be feasible, and this can not be fixed.

19.1: The Simplex Algorithm in Detail

Simplex algorithm

Simplex($\widehat{L}$ a LP)

Transform $\widehat{L}$ into slack form.

Let L be the resulting slack form.

$L' \leftarrow$ **Feasible**(L)

$x \leftarrow$ **LPStartSolution**(L')

$x' \leftarrow$ **SimplexInner**(L', x) (*)

$z \leftarrow$ objective function value of x'

if $z > 0$ **then**

return "No solution"

$x'' \leftarrow$ **SimplexInner**(L, x')

return x''

Simplex algorithm...

1. **SimplexInner**: solves a LP if the trivial solution of assigning zero to all the nonbasic variables is feasible.
2. $L' = \textbf{Feasible}(L)$ returns a new LP with feasible solution.
3. Done by adding new variable x_0 to each equality.
4. Set target function in L' to $\min x_0$.
5. original LP L feasible $\iff$ LP L' has feasible solution with $x_0 = 0$.
6. Apply **SimplexInner** to L' and solution computed (for L') by **LPStartSolution**(L').
7. If $x_0 = 0$ then have a feasible solution to L.
8. Use solution in **SimplexInner** on L.
9. need to describe **SimplexInner**: solve LP in slack form given a feasible solution (all nonbasic vars assigned value 0).

Notations

$\boldsymbol{B}$ - Set of indices of basic variables
$\boldsymbol{N}$ - Set of indices of nonbasic variables
$\boldsymbol{n = |N|}$ - number of original variables
$\boldsymbol{b, c}$ - two vectors of constants
$\boldsymbol{m = |B|}$ - number of basic variables (i.e., number of inequalities)
$\boldsymbol{A = \{a_{ij}\}}$ - The matrix of coefficients
$\boldsymbol{N \cup B = \{1, \ldots, n + m\}}$
$\boldsymbol{v}$ - objective function constant.
LP in slack form is specified by a tuple $\boldsymbol{(N, B, A, b, c, v)}$.

The corresponding LP

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \text{ for } i \in B, \\ & x_i \geq 0, \qquad \forall i = 1, \ldots, n+m. \end{aligned}$$

Reminder - basic/nonbasic

Nonbasic variables

$$
\begin{aligned}
\max \quad z &= 29 - \frac{1}{9}x_3 - \frac{1}{9}x_5 - \frac{2}{9}x_6 \\
x_1 &= 8 + \frac{1}{6}x_3 + \frac{1}{6}x_5 - \frac{1}{3}x_6 \\
x_2 &= 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6 \\
x_4 &= 18 - \frac{1}{2}x_3 + \frac{1}{2}x_5
\end{aligned}
$$

Basic variables

$\forall i \quad x_i \geq 0$

19.2: The SimplexInner Algorithm

The SimplexInner Algorithm

Description **SimplexInner** algorithm:

1. LP is in slack form.
2. Trivial solution $x = \tau$ (i.e., all nonbasic variables zero), is feasible.
3. objective value for this solution is v.
4. Reminder: Objective function is $z = v + \sum_{j \in N} c_j x_j$.
5. x_e: nonbasic variable with positive coefficient in objective function.
6. Formally: e is one of the indices of $\left\{ j \;\middle|\; c_j > 0, j \in N \right\}$.
7. x_e is the ***entering variable*** (enters set of basic variables).
8. If increase value x_e (from current value of 0 in τ)...
9. ... one of basic variables is going to vanish (i.e., become zero).

Choosing the leaving variable

1. x_e: ***entering variable***
2. x_l: ***leaving*** variable – vanishing basic variable.
3. increase value of x_e till x_l becomes zero.
4. How do we now which variable is x_l?
5. set all nonbasic to 0 zero, except x_e
6. $x_i = b_i - a_{ie}x_e$, for all $i \in B$.
7. Require: $\forall i \in B \quad x_i = b_i - a_{ie}x_e \geq 0$.
8. $\implies x_e \leq (b_i/a_{ie})$
9. $l = \arg\min_i b_i/a_{ie}$
10. If more than one achieves $\min_i b_i/a_{ie}$, just pick one.

Pivoting on $\mathbf{x_e}$...

1. Determined x_e and x_l.
2. Rewrite equation for x_l in LP.
   1. (Every basic variable has an equation in the LP!)
   2. $x_l = b_l - \sum_{j \in N} a_{lj} x_j$

$$\implies \qquad x_e = \frac{b_l}{a_{le}} - \sum_{j \in N \cup \{l\}} \frac{a_{lj}}{a_{le}} x_j, \qquad \text{where } a_{ll} = 1.$$

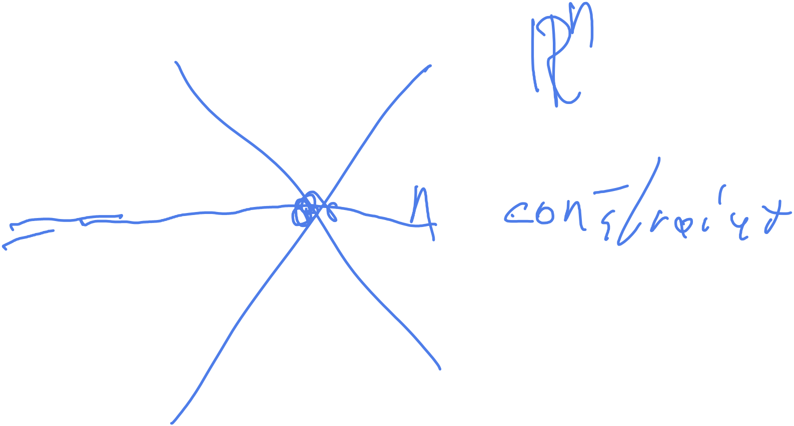
3. Cleanup: remove all appearances (on right) in LP of x_e.
4. Substituting x_e into the other equalities, using above.
5. Alternatively, do Gaussian elimination remove any appearance of x_e on right side LP (including objective).
   Transfer x_l on the left side, to the right side.

Pivoting continued...

1. End of this process: have new *equivalent* LP.
2. basic variables: $B' = (B \setminus \{l\}) \cup \{e\}$
3. non-basic variables: $N' = (N \setminus \{e\}) \cup \{l\}$.
4. End of this ***pivoting*** stage:
   LP objective function value increased.
5. Made progress.
6. LP is completely defined by which variables are basic, and which are non-basic.
7. Pivoting never returns to a combination (of basic/non-basic variable) already visited.
8. ...because improve objective in each pivoting step.
9. Can do at most $\binom{n+m}{n} \leq \left(\frac{n+m}{n} \cdot e\right)^n$.
10. examples where 2^n pivoting steps are needed.

Simplex algorithm summary...

1. Each pivoting step takes polynomial time in n and m.
2. Running time of **Simplex** is exponential in the worst case.
3. In practice, **Simplex** is extremely fast.



Pivoting with zeroes?

Clicker question

Consider a pivoting step, with x_e as the entering variable, and x_ℓ as the leaving variable, with the relevant constraint in the LP being:

$$x_\ell = 0 - \sum_{j \in N} a_{lj} x_j.$$

1. Doing the pivoting step would involve division by zero, and as such the **Simplex** algorithm would fail.
2. There is no problem.
3. In an LP the constant in a constraint can never be zero, so this is an impossible scenario.
4. If there is any problem, it can be solved by choosing a different entering/leaving variables.
5. The pivoting step would not improve the LP objective function. **Simplex** might pivot in a loop forever.

Degeneracies

1. **Simplex** might get stuck if one of the b_is is zero.
2. More than $> m$ hyperplanes (i.e., equalities) passes through the same point.
3. Result: might not be able to make any progress at all in a pivoting step.
4. Solution I: add tiny random noise to each coefficient.
   Can be done symbolically.
   Intuitively, the degeneracy, being a local phenomena on the polytope disappears with high probability.

Degeneracies – cycling

1. Might get into cycling: a sequence of pivoting operations that do not improve the objective function, and the bases you get are cyclic (i.e., infinite loop).
2. Solution II: ***Bland's rule***.
   Always choose the lowest index variable for entering and leaving out of the possible candidates.
   (Not prove why this work - but it does.)

19.2.1: Correctness of linear programming

Correctness of LP

Definition

A solution to an LP is a ***basic solution*** if it the result of setting all the nonbasic variables to zero.

Simplex algorithm deals only with basic solutions.

Theorem

For an arbitrary linear program, the following statements are true:

1. *If there is no optimal solution, the problem is either infeasible or unbounded.*
2. *If a feasible solution exists, then a basic feasible solution exists.*
3. *If an optimal solution exists, then a basic optimal solution exists.*

Proof: is constructive by running the simplex algorithm.

19.2.2: On the ellipsoid method and interior point methods

On the ellipsoid method and interior point methods

1. **Simplex** has exponential running time in the worst case.
2. ***ellipsoid method*** is *weakly* polynomial.
   It is polynomial in the number of bits of the input.
3. Khachian in 1979 came up with it. Useless in practice.
4. In 1984, Karmakar came up with a different method, called the *interior-point method*.
5. Also weakly polynomial. Quite useful in practice.
6. Result in arm race between the interior-point method and the simplex method.
7. BIG OPEN QUESTION: Is there *strongly* polynomial time algorithm for linear programming?

Solving LPs without ever getting into a loop - symbolic perturbations

Details in the class notes.

Notes

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