11/6/18

Linear programming

Low dimension 2, 3, 4,

$x_1, x_2, \dots, x_d$ d

$$\sum_{i=1}^{d} a_i^j x_i \le c^j \qquad j = 1, \dots, m$$

$\forall_{i,j}$ $a_{i,j} \in \mathbb{R}$ $c_j \in \mathbb{R}$

$\exists (x_1, \dots, x_d)$ s.t. all inequalities hold

Find solution s.t. some linear expression is maximized.

$$\max \sum c_i x_i$$

$x_1, \dots, x_d$: items

Budget

x_1: # of F35 airplanes you are going to buy

x_2: # of people on social security

$$5 \cdot 10^8 x_1 + 10000 \cdot x_2 \cdots + \le 2.3 \cdot 10^{12}$$

$X_i \geq 100$

$\boxed{\sum DoD} \leq 700 \cdot 10^9$

Network flow Linear Program

- Linear program
  + all variables are integers
  $\Rightarrow$ Integer programming (NPC)

$x_i \in \{0,1\}$ $0 \leq x_i \leq 1$

x_i $1-x_i$

$(x \vee y \vee z)$ $x+y+z \geq 1$

Linear programming

n variables and m constraints

$O(m^n)$ solving LP using the simplex algorithm

Tabulu method

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early 80 Ellipsoid method

Polynomial time in the number of bits.

Open Question

Can LP be solve in strongly polynomial time?

History

Kantorovich 1934n

Danzig ←

2D imensions

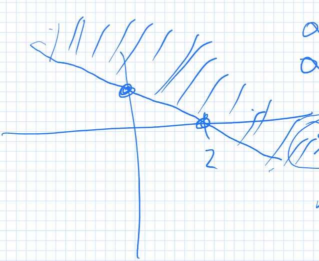
$ax+by \geq c$

$a'x+b'y \geq c'$

$\vdots$

$x+2y \geq 2$

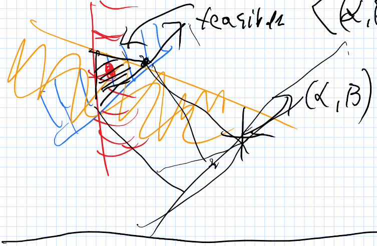
$x+2y = 2$



$ax+by \overset{>}{\underset{<}{=}} c$

$\max \alpha x+\beta y$

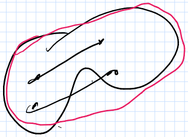
$\langle(\alpha,\beta),(x,y)\rangle$



$\|(\alpha,\beta)\| = 1$

Convex set

$\mathbb{R}^d$

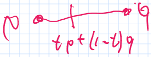


Convex combination of points
$p_1, \dots, p_m \in \mathbb{R}^d$

$q = \sum \alpha_i p_i$ ← convex combination

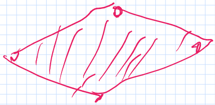
$\alpha_i \in [0,1]$

$\sum \alpha_i = 1$



Convex hull of a set X is

$$CH(X) = \left\{ \sum \alpha_i p_i \;\middle|\; \begin{array}{l} p_i \in X \\ \alpha_i \geq 0 \\ \sum \alpha_i = 1 \end{array} \right\}$$



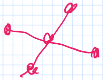
Radon theorem

$X \subseteq \mathbb{R}^d$ $\quad d = 2$

$|X| \geq d+2$ $\quad 4$

$\Rightarrow X = Y \cup Z$

s.t. $CH(Y) \cap CH(\ ) \neq \emptyset$

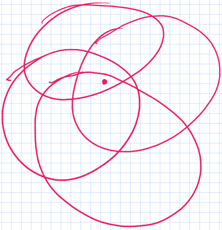


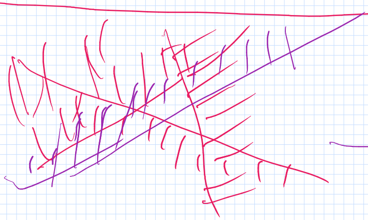
Helly's theorem

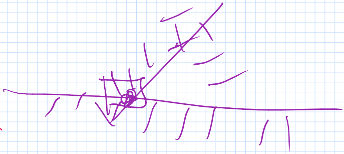
$C_1, C_2, \ldots, C_m \subseteq \mathbb{R}^d$ convex

$\bigcap_{i=1}^{m} C_i \neq \emptyset \iff$ For any subset of $d+1$ sets $\mathcal{Y} \subseteq \{C_1, \ldots, C_m\}$

$\bigcap_{Y \in \mathcal{Y}} Y \neq \emptyset$







d variables

d inequalities

d equations

d variables

$$\rightarrow a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_d = c_1$$

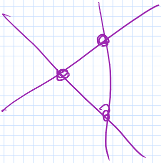
$$\vdots$$

$$\rightarrow a_{d,1}x_1 + \cdots a_{d,d}x_d = c_d$$

Gausian elimination

$d^2 \cdot d$ $O(d^3)$ Good! polynomial

$d+1$ constraints that are not feasible



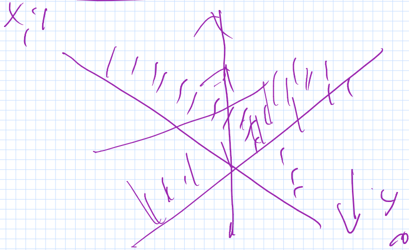
$$\binom{d+1}{d} = d+1$$

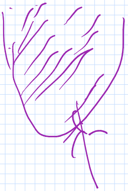
$$O(d^4)$$

Stupidest LP solver

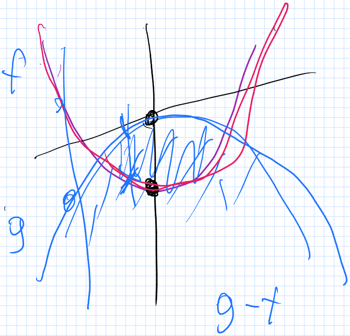
m constraints

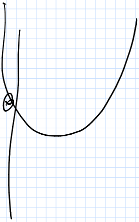
$$\binom{m}{d} O(d^3 + \underbrace{md}) = O(m^{d+1})$$

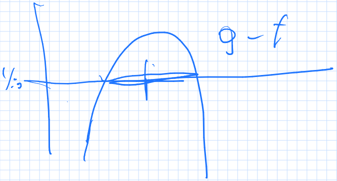




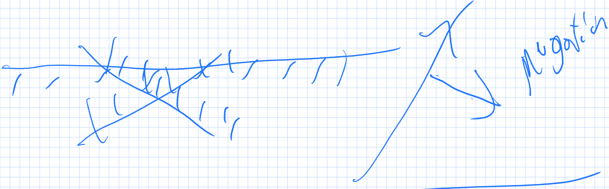
coordinate polytope







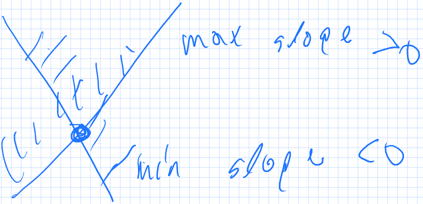
Randomized algorithm



$h_1, \ldots, h_n$ half planes

Find lowest point

- H^c = all halfplane above their bounding lines



$h_1, h_2,$ random permutation of remaining constraints

$p_i = LPSol(h_1, \ldots, h_i)$

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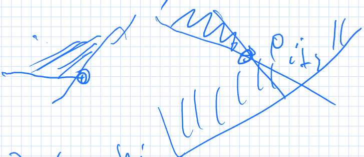
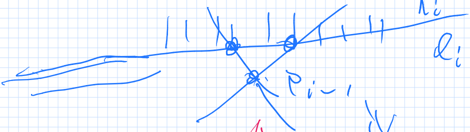
p_{i-1}

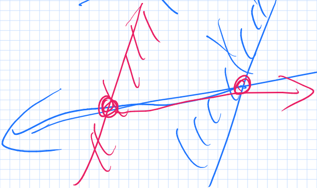


$p_{i-1} \in h_i$ $\quad p_i \leftarrow p_{i-1}$

$\downarrow 0$

$p_{i-1} \notin h_i$





$i-1$

$O(i)$

$$\sum_{i=3}^{n}\left(O(i)+\frac{2}{i}O(i)\right)=O(n)$$