# 11/1/18) Network flow

## Ford Fulkerson Method

- Integrality.

$$O(|f^*|m)$$

**Idea:** Augment along shortest path.

## Edmonds-Karp alg

Shortest path = min number of hops.

**Claim**

For a flow $f$, let $G_f$ be the residual network

$\delta_f(v)$: shortest path from $s$ to $v$ in $G_f$

$\delta_f(v)$ is non-decreasing as EK runs. $\forall v \in V$

**Lemma**

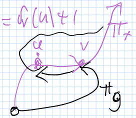
An edge $u \to v$ can disappear and reappear at most $n/2$ times during EK alg execution

**Proof**

$f$: flow just before $u \to v$ dis

$g$: flow just before $u \to v$ reappear

$\delta_f(v) = \delta_f(u) + 1$



$$\delta_g(u) = \delta_g(v) + 1 \geq \underset{\underset{\delta_f(u)+1}{\parallel}}{\delta_f(v)} + 1 = \delta_f(u) + 2$$

$$\frac{n}{2}$$

∎

**Lemma** EK does $\frac{mn}{2}$ iterations,

**Proof** $\frac{mn}{2}$ ∎

**Running time** of EK is

$$\boxed{O(nm^2)}$$

$$O\left(\frac{nm}{2} m\right)$$

## Proof of claim

Proof by contradiction.

There exist a vertex $v$ s.t.

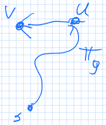
$$\delta_f(v) > \delta_g(v)$$

$f$ flow before an iteration

$g$ flow after iteration

- Look on first iteration where this happens.
- $v$ is the vertex minimizing $\delta_g(v)$

$\Pi_g$: shortest path from $s$ to $v$ in $G_g$.

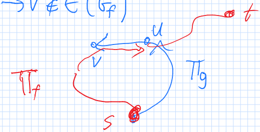


- $u \to v \in E(G_f)$

$$\delta_g(u) = \delta_g(v) - 1 < \delta_f(v) - 1 \le \delta_f(u) + 1 - 1$$
$$= \delta_f(u)$$

$$\boxed{\delta_g(u) < \delta_f(u)} \qquad \boxed{\delta_g(u) = \delta_g(v) - 1}$$

- $u \to v \notin E(G_f)$



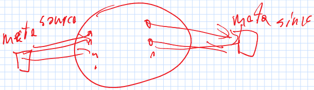
$u \to v$ reappeared $f$ "$\to$" $g$

$$\delta_f(u) = \delta_f(v) + 1 > \delta_g(v) + 1$$
$$> \delta_g(v)$$
$$= \delta_g(u) + 1$$

$\delta_f(u) > \delta_g(u)$ A contradiction. ▨

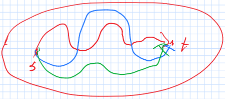
## Network flow applications

## Multiple sources sinks



## Edge disjoint paths

G directed



K

Task: Compute k edge disjoint paths from s to t in G

- Set capacity of all edges to be 1

Compute max flow from s to t in G.

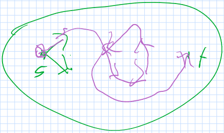
assume $|f| \geq K$

**Claim** There are $|f|$ edge disjoint paths from s to t.

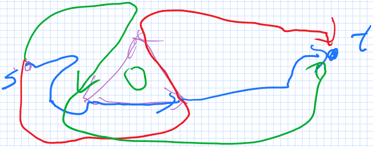
**Proof** Assume f is integral

0/1-flow

G'



$K = |f|$



# Menger's theorem

Let k be the maximum number of edge disjoint paths from s to t.

- t = minimum number of edges one has to remove s.t. s and t get disconnected.

$$\boxed{t = k}$$

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**Proof:** Min-cut max-flow.

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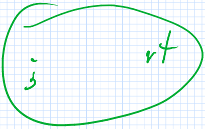
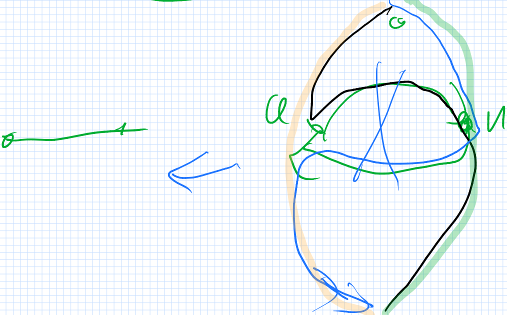
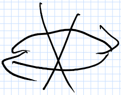
## k-edge disjoint in undirected graph





- max flow



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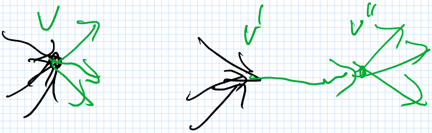
Vertex disjoint paths directed case



$n + m$

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## Circulations with lower bounds

G, capacities

for every edge there is a lower bound $\ell(e) \leq c(e)$.

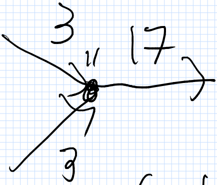
task: Is there a flow/circulation that conserve flow at each vertex and $\forall e \in E$

$$c(e) \geq f(e) \geq \ell(e)$$

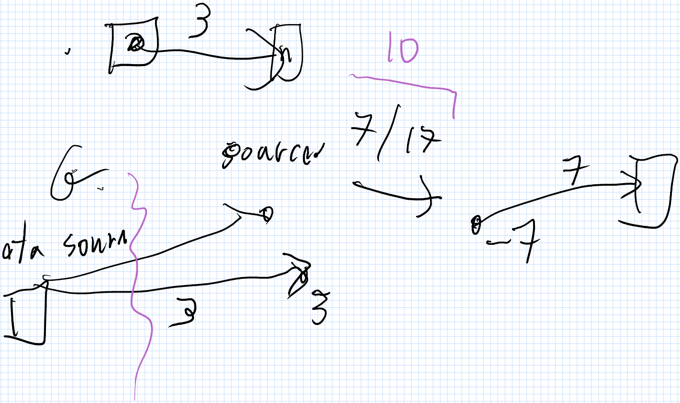
Fake flow

$g(e) = \ell(e)$

- everything is good.



$6 - 17 = -11$



h flow

gth required flow