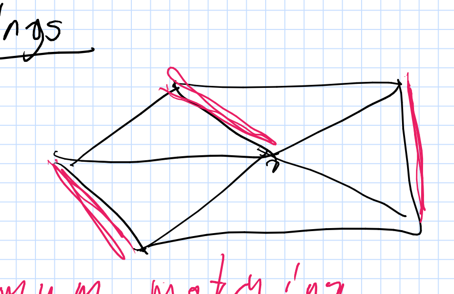
473

Matchings

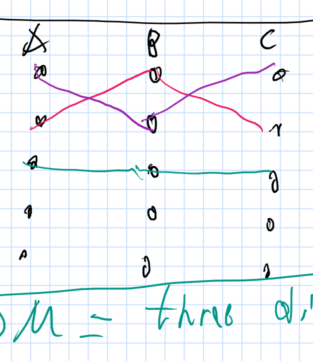
G



maximum matching
maximum cardinality matching

Max weight matching ✓

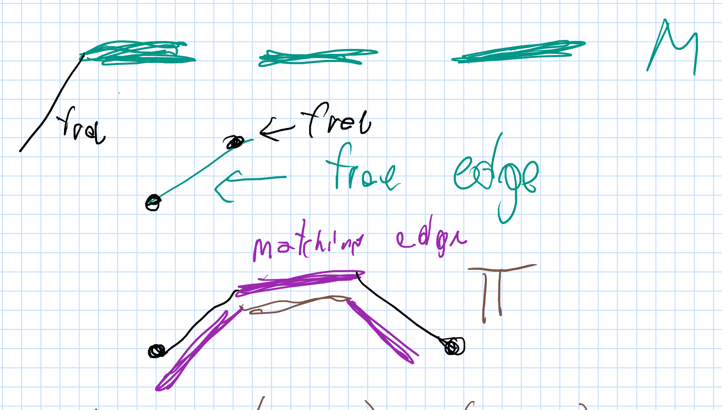
	Bipartite	non Bipartite
unw	✓	✓
weighted	✓	



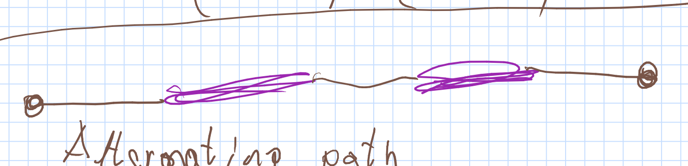
n vertices

NP-Hard

3DM = three dimensional matching



$M \oplus \pi = (M \setminus \pi) \cup (\pi \setminus M)$



Alternating path

Augmenting (path) — simple

– Alternating path

+ starting and ending in a free vertex.

Claim

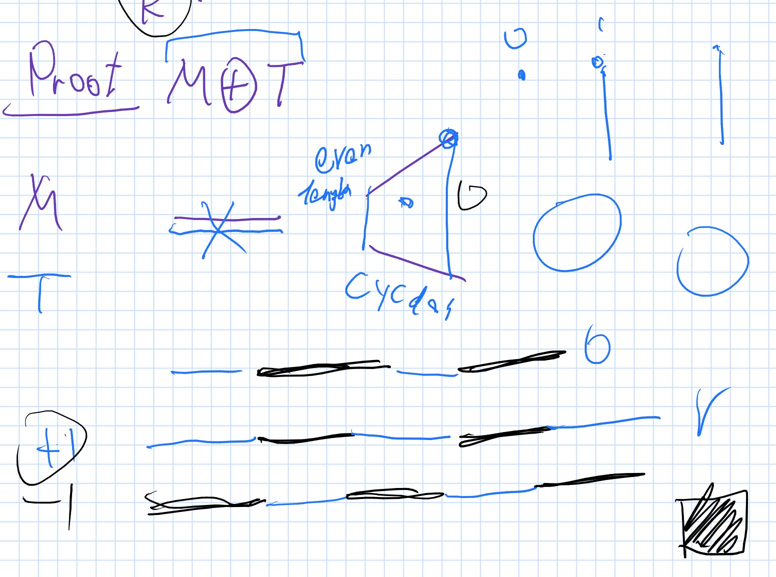
M is a matching

T is a maximum matching

$k = |T| - |M|$

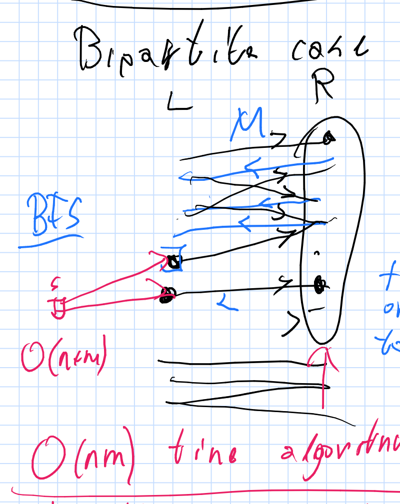
$\exists k$ disjoint augmenting paths of M, and one of them has at most $\leq \frac{n}{k}$.

Proof $M \oplus T$



How to compute an augmenting path

Bipartite case



alternating

shortest path

free vertex on left to free vertex on right.

$O(nm)$ time algorithm

Hopcroft Karp algorithm

$O(nm)$

$O(m\sqrt{n})$

Claim If always augmenting along shortest path then the length of the paths do not decrease.

$\pi_1, \pi_2, \ldots, \pi_k$

$|\pi_1| \leq |\pi_2| \leq \ldots \leq |\pi_k|$

Claim

Consider a block of augmenting paths

$\pi_1, \pi_2, \ldots \pi_i, \pi_{i+1}, \ldots \pi_{j}, \ldots \pi_k$ — all of the same size

The augmenting paths in the block are disjoint!

– For $k = 1$ to $\sqrt{n}$ do

M: current matching

$\Pi_k \leftarrow$ Find max collection of augmenting paths of len $2k+1$

$M \leftarrow M \oplus \Pi_k$ $O(m\sqrt{n})$ $O(m\sqrt{n})$

– Continue running the old alg

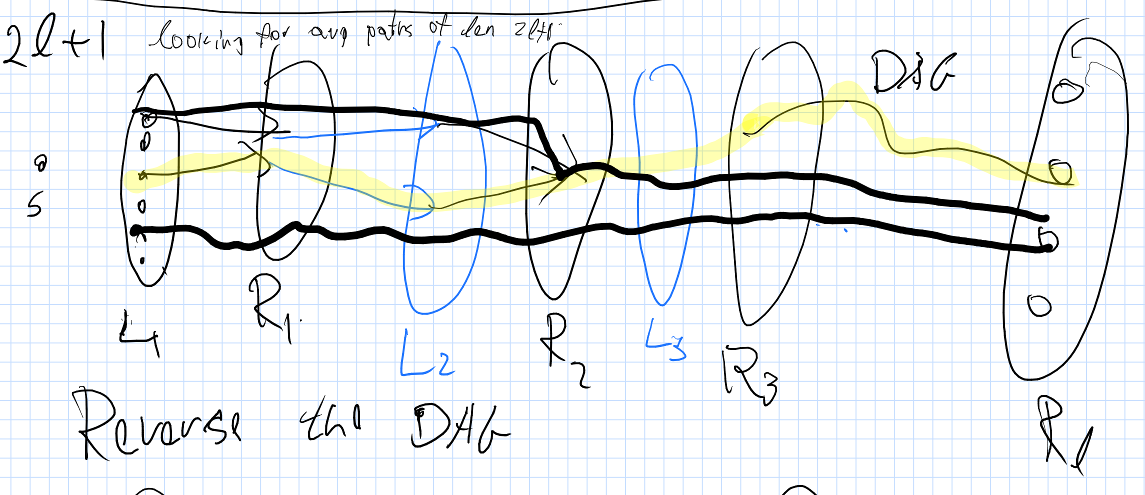
Lemma

$|opt \oplus M|$ $k = |opt| - |M|$

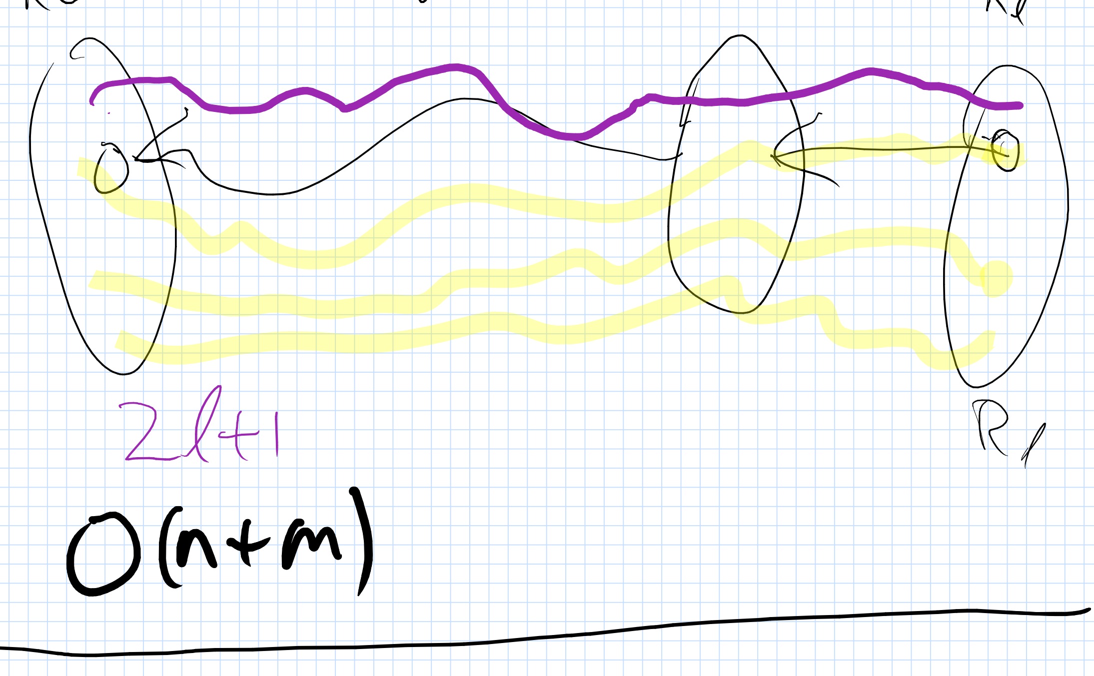
$\exists$ aug path of len $\leq \frac{n}{k}$

$\frac{n}{k} \geq \sqrt{n} \Rightarrow k \leq \sqrt{n}$

$2l+1$ Looking for aug paths of len $2l+1$



Reverse the DAG



$O(n+m)$

Lemma

M matching

π shortest augmenting path for M

$M' = M \oplus \pi$

π' shortest aug path for M'

$|\pi'| \geq |\pi| + 2|\pi \cap \pi'|$

Proof

$N = M \oplus \pi \oplus \pi'$

$|N| = |M| + 2$

$N \ominus M$ = contains two disjoint aug paths for M

σ_1, σ_2 be these two paths

$|\sigma_1| \geq |\pi|$ $|\sigma_2| \geq |\pi|$

$|N \ominus M| \geq |\sigma_1| + |\sigma_2| \geq 2|\pi|$

$N \oplus M = N \oplus (N \oplus \pi_1 \oplus \pi_2)$

$= \pi + \pi'$

$|\pi| + |\pi'| - 2|\pi \cap \pi'| = \oplus \pi'|$

$|\pi| + |\pi'| - 2|\pi \cap \pi'| = |M \oplus N|$

$\geq |\sigma_1| + |\sigma_2| \geq 2|\pi|$

$\Rightarrow |\pi'| \geq |\pi| - 2|\pi \cap \pi'|$