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- Closest pair
- Quick Sort

Backward Analysis

$\alpha_1, \alpha_2, \dots, \alpha_n$ real numbers

Randomly permuted

$m_i = \min(\alpha_1, \alpha_2, \dots, \alpha_i)$

$m_1, m_2, \dots, m_n$

$X_i = 1 \iff m_i < m_{i-1}$

$P[X_i = 1] = \frac{1}{i}$

$$E\left[\sum X_i\right] = \sum E[X_i]$$

$$= \sum P[X_i = 1]$$

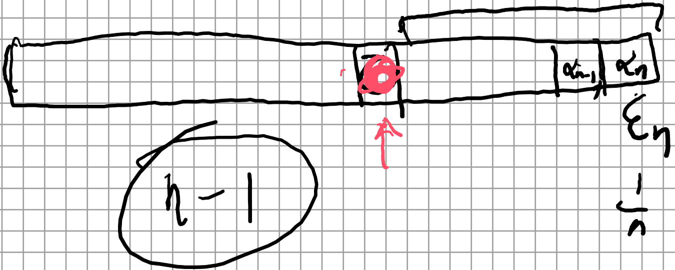
$$= \sum_{i=1}^{n} \frac{1}{i} \leq \ln n + 1$$

$\mathcal{E}_i$ the event that

$X_i = 1$ (i.e. $m_i < m_{i-1}$)

$$P[\mathcal{E}_i] = \frac{1}{i}$$

<u>Claim</u> The events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are independent.



$X_1, X_2, \ldots, X_n$

$t = 80$

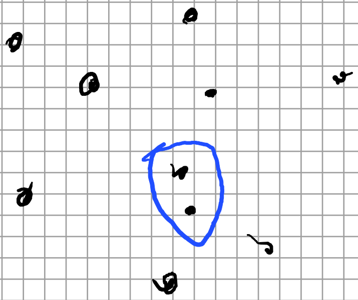
Chernoff ineq

$X_1, X_2, \ldots, X_n \in \{0,1\}$ independent

$\mu = E[\sum X_i]$ $\quad P[\sum X_i > t\mu] \leq e^{-\frac{t\mu}{8}} \leq \frac{1}{n^{10}}$

$t \geq 1$

Closest Pair



(x_i, y_i)

$i = 1, \dots, n$

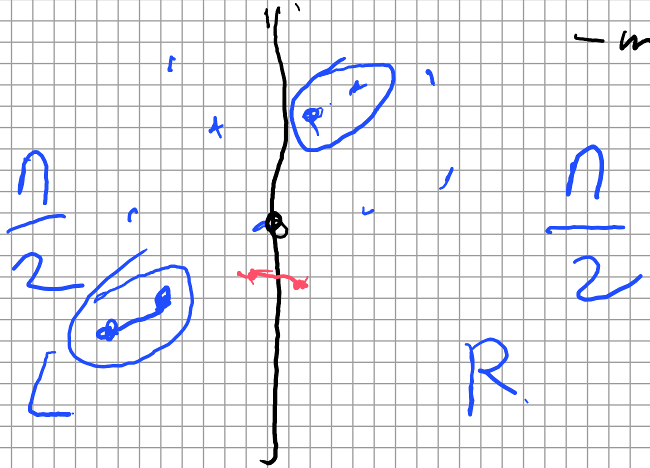
$O(n^2)$

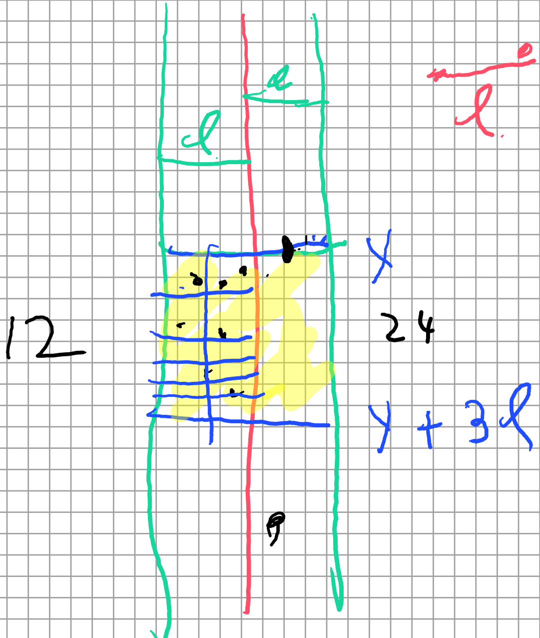
$O(n \log n)$ divide and conquer

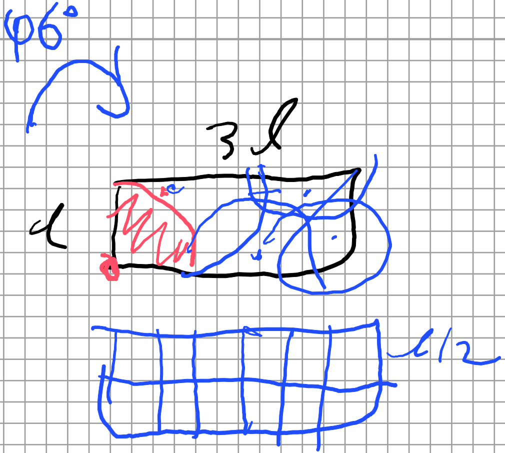
$O(n)$

CP in $O(n \log n)$ time

- median by x-axis




ℓ
ℓ
ℓ
y
24
12
y + 3ℓ


φ60°
3ℓ
ℓ
ℓ/2

$$T(n) = O(n) + 2T\left(\frac{n}{2}\right) + O(n \log n)$$

$$T(n) = O(n \log^2 n)$$

- Pre sort the points

$$T(n) = O(n) + 2T(n/2) = O(n \log n)$$

Uniqueness

$x_1, x_2, \ldots, x_n$ numbers

Q: Are they all unique?

Sort + scan $O(n \log n)$ time

Theorem Uniqueness takes $\Omega(n \log n)$ time in the comp model

Different model of computation

- Hashing (careful). $O(1)$
- floor function

$$\left\lfloor \frac{x}{y} \right\rfloor \text{ integer part}$$

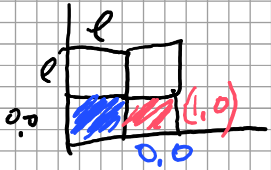
- Uniform RAM model

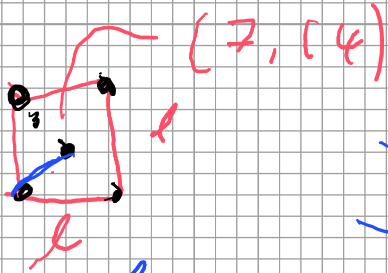
Verification

ℓ, d **Claim**: $\ell = CP(\ell)$

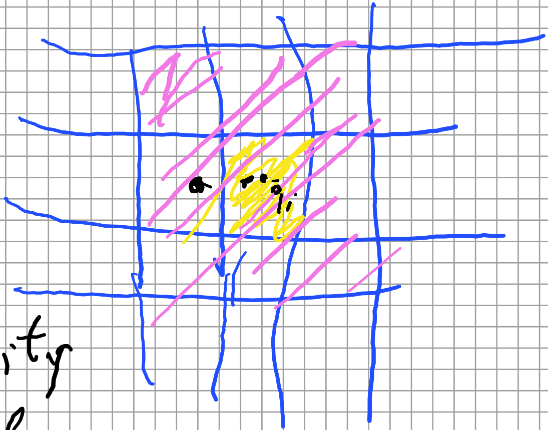
Grid

$$(x, y) \longrightarrow \left(\left\lfloor \frac{x}{\ell} \right\rfloor, \left\lfloor \frac{y}{\ell} \right\rfloor \right)$$





$$\sqrt{2}\,\frac{\ell}{2} = \frac{\ell}{\sqrt{2}}$$



We can verify $CP(P) = \ell$ in $O(n)$ time.

Randomized incremental construction of CP

input: $p_1, \ldots, p_n$

$q_1, q_2, \ldots, q_n$: random permutation

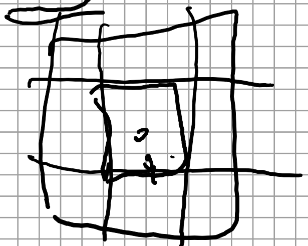
$\ell_2 = \|q_1 q_2\|$

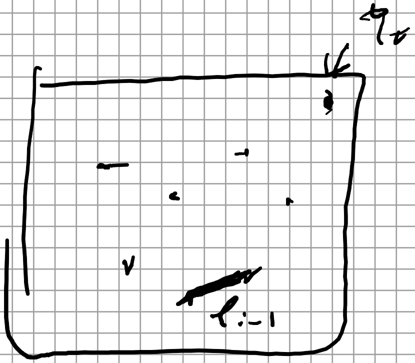
For $i = 3$ to n do

$\ell_i = CP(q_1, \ldots, q_i)$

$\ell_i = \ell_{i-1}$?

$\ell_i < \ell_{i-1}$ $\quad O(1)$



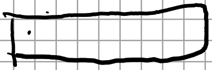


$q_1, \ldots, q_{i-1}$

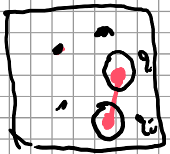
If $\ell_i < \ell_{i-1}$ then rebuild verification DS for $q_1 \ldots q_i$. $\quad O(\ell)$

$$\sum_{i=1}^{n} O(i) = O(n^2)$$

$$\Pr[\ell_i < \ell_{i-1}] \le \frac{2}{i}$$



$q_1, q_2, \dots, q_i$



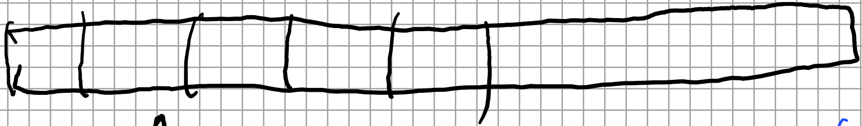
Expected work in ith iteration

$$E[R_i] = O(1) + P[l_i < l_{i-1}]\, O(i)$$

$$\leq O(1) + \frac{2}{i}\, O(i)$$

$$= O(1),$$

$$E\left[\sum R_i\right] = O(n)$$

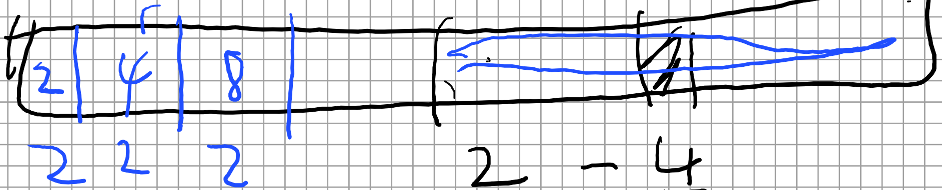


$$\sum_{i=2}^{n} \frac{2}{i} = 2\ln n$$

$n/2$

$$\frac{4}{n} \cdot \frac{n}{2} = 2$$

$n/2$ n



$$\frac{2}{n/2} = \frac{4}{n}$$