CS 473, Fall 2017
Homework 9 (due Nov 29 Wednesday at 8pm)

You may work in a group of at most 3 students. Carefully read `http://engr.course.illinois.edu/cs473/policies.html` and `http://engr.course.illinois.edu/cs473/integrity.html`. One member of each group should submit via Gradescope.

1. [*16 pts*]

   (a) [*5 pts*] Convert the following optimization problems into a decision problem and show that the corresponding decision problem is in NP.

   Input: a set S of n numbers $a_1, \ldots, a_n$ and an integer $k \leq n$.

   Output: a partition of S into k subsets $S_1, \ldots, S_k$ minimizing the cost of the partition, which is defined as

   $$\sum_{i=1}^{k} \left(\sum_{j \,:\, a_j \in S_i} a_j \right)^2 .$$

   [Note: the subsets don't have to be contiguous blocks, unlike the question from midterm 1!]

   (b) [*3 pts*] Show that if we could solve your decision problem in (a) in time polynomial in the number of input bits, then we could also compute the minimum cost in time polynomial in the number of input bits.

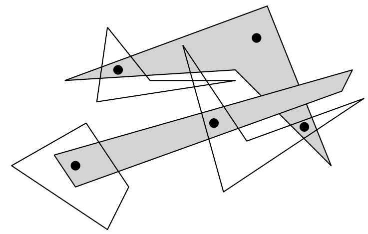
   (c) [*8 pts*] Show that if we could solve your decision problem in (a) in time polynomial in the number of input bits, then we could also compute an optimal partition into subsets $S_1, \ldots, S_k$ in time polynomial in the number of input bits. [Hint: call the subroutine in part (b) $O(n^2)$ times...]

2. [*24 pts*] A quadrilateral is a polygon with 4 vertices (which may or may not be convex). Consider the following problem called Quad-Cover:

   Input: a set P of m points, a set Q of n quadrilaterals in 2D, and an integer k.

   Output: "yes" iff there exists a subset $T \subseteq Q$ of at most k quadrilaterals such that every point in P lies inside some quadrilateral in T.

   In the following example, the answer is "yes" for $k = 2$ (with the optimal subset shaded).



(a) [*4 pts*] Prove that QUAD-COVER is in NP.

(b) [*17 pts*] Prove that QUAD-COVER is NP-complete via a polynomial-time reduction based on VERTEX-COVER.

(Hint: Given a graph $G = (V, E)$ where $V = \{1, \ldots, n\}$, to construct the set P of points, map each $ij \in E$ $(i < j)$ to a point (i, j). For the set Q, you will need to define certain non-convex quadrilaterals to cover particular rows/columns somehow. . .)

(c) [*3 pts*] Illustrate your reduction for the graph G shown below, with $k = 3$. Draw the set P of points and the set Q of quadrilaterals in your construction. Also draw the solution T that corresponds to the vertex cover $S = \{2, 4, 6\}$ of G.