CS 473, Fall 2017
Homework 10 (due Dec 6 Wednesday at 8pm)

You may work in a group of at most 3 students. Carefully read `http://engr.course.illinois.edu/cs473/policies.html` and `http://engr.course.illinois.edu/cs473/integrity.html`. One member of each group should submit via Gradescope.

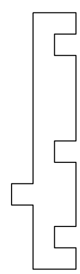
1. [*20 pts*] Prove that the following problem "JIGSAW-PUZZLE" is NP-complete:

   Input: a set P of n polygons where all vertices have integer coordinates and all edges are horizontal or vertical, and $M, N \leq 100n$.

   Output: yes iff there exists a placement of the n polygons so that they do not overlap and their union is exactly the $M \times N$ rectangle.

   For simplicity, you may assume that we are only allowed to place each polygon by translation only, but *not rotation* (not even reflection).

   Hint: Reduce from HAMILTONIAN-PATH (a variant of the HAMILTONIAN-CYCLE), which you may assume is NP-complete. Pieces that look like the following may help:



2. [*25 pts*] In this problem, you will investigate the minimum vertex cover problem in the special case when the input graph has maximum degree 3. (This special case is still NP-hard.) Although the natural greedy algorithm does not give good approximation in general, you will show that it gives approximation factor strictly better than 2 in this special case.

   (a) [*5 pts*] First, show that for graphs with maximum degree at most 2, the minimum vertex cover problem can be solved exactly in polynomial time.

   (b) [*3 pts*] Next, consider the following greedy algorithm for a given graph G of maximum degree 3:

   $A = \emptyset$
   while there is a vertex v of degree 3 do
       insert v to A, and remove v from the graph
   let B be the set of remaining vertices
   compute an exact minimum vertex cover S_B for the subgraph G_B formed by B
   return $S = A \cup S_B$

Show that this algorithm returns a vertex cover S and runs in polynomial time.

(c) [*4 pts*] Let S^* be the minimum vertex cover. Let $A^* = A \cap S^*$. For each $i \in \{0, 1, 2\}$, let B_i^* be the subset of vertices of $B \cap S^*$ that have degree i in the subgraph G_B.
Prove that $|S| \leq |A| + |B_1^*| + |B_2^*|$.

(d) [*4 pts*] Prove that each vertex in $A - A^*$ is adjacent to exactly 3 vertices, all of which are in $B_0^* \cup B_1^* \cup B_2^*$.

(e) [*4 pts*] Using (d), prove that $3(|A| - |A^*|) \leq 3|B_0^*| + 2|B_1^*| + |B_2^*|$.

(f) [*4 pts*] Using (c) and (e), prove that $|S| \leq |A^*| + |B_0^*| + (5/3)|B_1^*| + (4/3)|B_2^*|$.

(g) [*1 pts*] Conclude that the above algorithm has approximation factor at most $5/3$.