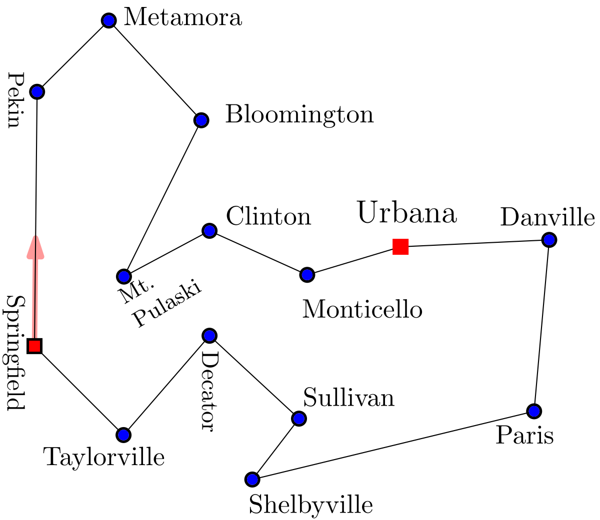
Approximation Algorithms for TSP

Lecture 26

Dec 2, 2016

Lincoln's Circuit Court Tour



Traveling Salesman/Salesperson Problem (TSP)

Perhaps the most famous discrete optimization problem

Input: A graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}_+$.
Goal: Find a Hamiltonian Cycle of minimum total edge cost

Graph can be undirected or directed. Problem differs substantially.
We will first focus on undirected graphs.

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Assumption for simplicity: Graph $G = (V, E)$ is a complete graph. Can add missing edges with infinite cost to make graph complete.

Observation: Once graph is complete there is always a Hamiltonian cycle but only Hamiltonian cycles of finite cost are Hamiltonian cycles in the original graph.

Important Special Cases

Metric-TSP: $G = (V, E)$ is a complete graph and c defines a metric space. $c(u, v) = c(v, u)$ for all u, v and $c(u, w) \leq c(u, v) + c(v, w)$ for all u, v, w.

Geometric-TSP: V is a set of points in some Euclidean d-dimensional space $\mathbb{R}^d$ and the distance between points is defined by some norm such as standard Euclidean distance, L_1/Manhatta distance etc.

Another interpretation of Metric-TSP: Given $G = (V, E)$ with edges costs c, find a tour of minimum cost that visits all vertices but can visit a vertex more than once.

Inapproximability of TSP

Observation: In the general setting TSP does not admit any bounded approximation.

- Finding or even deciding whether a graph $G = (V, E)$ has Hamiltonian Cycle is NP-Hard
- Alternatively, suppose $G = (V, E)$ is a simple graph that we complete with infinite cost edges. If G has a Hamilton Cycle then there is a TSP tour of cost n else it is cost ∞.

Metric-TSP

Metric-TSP is simpler and perhaps a more natural problem in some settings.

Theorem

Metric-TSP is NP-Hard.

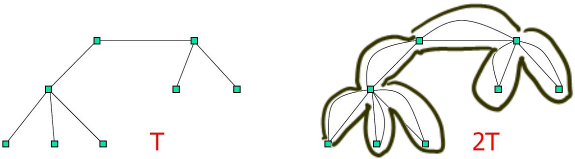
Proof.

Given $G = (V, E)$ we create a new complete graph $G' = (V, E')$ with the following costs. If $e \in E$ cost $c(e) = 1$. If $e \in E' - E$ cost $c(e) = 2$. Easy to verify that c satisfies metric properties. Moreover, G' has TSP tour of cost n iff G has a Hamiltonian Cycle. □

Approximation for Metric-TSP

MST-Heuristic$(G = (V, E), c)$

Compute an minimum spanning tree (MST) T in G
Obtain an Eulerian graph $H = 2T$ by doubling edges of T
An Eulerian tour of H gives a tour of G
Obtain Hamiltonian cycle by shortcutting the tour



Analyzing MST-Heuristic

Lemma

Let $c(T) = \sum_{e \in T} c(e)$ *be cost of MST. We have* $c(T) \leq OPT$.

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A TSP tour is a connected subgraph of G and MST is the cheapest connected subgraph of G. □

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Theorem

MST-Heuristic gives a 2*-approximation for Metric-TSP.*

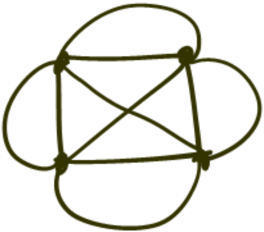
Proof.

Cost of tour is at most $2c(T)$ and hence MST-Heuristic gives a 2-approximation. □

Background on Eulerian graphs

Definition

An *Euler tour* of an undirected multigraph $G = (V, E)$ is a closed walk that visits each edge exactly once. A graph is Eulerian if it has an Euler tour.



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Theorem (Euler)

An undirected multigraph $G = (V, E)$ *is Eulerian iff* G *is connected and every vertex degree is even.*

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Theorem (Euler)

An undirected multigraph $G = (V, E)$ is Eulerian iff G is connected and every vertex degree is even.

Theorem

A directed multigraph $G = (V, E)$ is Eulerian iff G is weakly connected and for each vertex v, $indeg(v) = outdeg(v)$.

Improved approximation for Metric-TSP

How can we improve the MST-heuristic?

Observation: Finding optimum TSP tour in G is same as finding minimum cost Eulerian subgraph of G (allowing duplicate copies of edges).

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Christofides-Heuristic$(G = (V, E), c)$

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Add edges to T to make Eulerian graph H
An Eulerian tour of H gives a tour of G
Obtain Hamiltonian cycle by shortcutting the tour
```

How do we edges to make T Eulerian?

Christofides Heuristic: 3/2 approximation

Christofides-Heuristic$(G = (V, E), c)$

```
Compute an minimum spanning tree (MST) T in G
Let S be vertices of odd degree in T (Note:  |S| is even)
Find a minimum cost matching M on S in G
Add M to T to obtain Eulerian graph H
An Eulerian tour of H gives a tour of G
Obtain Hamiltonian cycle by shortcutting the tour
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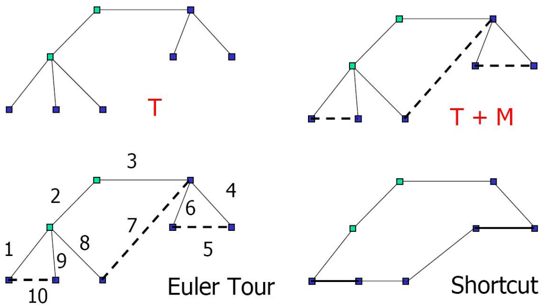


Analysis of Christofides Heuristic

Main lemma:

Lemma

$c(M) \leq OPT/2$.

Assuming lemma:

Theorem

Christofides heuristic returns a tour of cost at most $3OPT/2$.

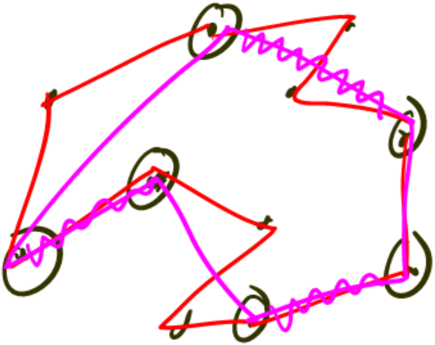
Proof.

$c(H) = c(T) + c(M) \leq OPT + OPT/2 \leq 3OPT/2$. Cost of tour is at most cost of H. □

Analysis of Christofides Heuristic

Lemma

Suppse $G = (V, E)$ *is a metric and* $S \subset V$ *be a subset of vertices. Then there is a TSP tour in* $G[S]$ *(the graph induced on* S*) of cost at most* OPT*.*



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Proof.

Let $C = v_1, v_2, \ldots, v_n, v_1$ be an optimum tour of cost OPT in G and let $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ where, without loss of generality $i_1 < i_2 \ldots < i_k$. Then consider the tour $C' = v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_{i_1}$ in $G[S]$. The cost of this tour is at most cost of C by shortcutting. □

Proof of lemma for Christofides heuristic

Lemma

$c(M) \leq OPT/2$.

Recall that M is a matching on S the set of odd degree nodes in T. Recall that $|S|$ is even.

Proof.

From previous lemma, there is tour of cost OPT for S in $G[S]$. Wlog let this tour be $v_1, v_2, \ldots, v_{2k}, v_1$ where $S = \{v_1, v_2, \ldots, v_{2k}\}$. Consider two matchings M_a and M_b where
$M_a = \{(v_1, v_2), (v_3, v_4), \ldots, (v_{2k-1}, v_{2k})$ and
$M_b = \{(v_2, v_3), (v_4, v_5), \ldots, (v_{2k}, v_1)$.
$M_a \cup M_b$ is set of edges of tour so $c(M_a) + c(M_b) \leq OPT$ and hence one of them has cost less than $OPT/2$. □

Other comments

Christofides heuristic has not been improved since 1976!
Major open problem in approximation algorithms.

For points in any fixed dimension d there is a polynomial-time approximation scheme. For any fixed $\epsilon > 0$ a tour of cost $(1+\epsilon)OPT$ can be computed in polynomial time. [Arora 1996, Mitchell 1996].

Excellent practical code exists for solving large scale instances of TSP that arise in several applications. See Concorde TSP Solver by Applegate, Bixby, Chvatal, Cook.

Directed Graphs and Asymmetric TSP (ATSP)

Question: What about directed graphs?

Equivalent of Metric-TSP is Asymmetric-TSP (ATSP)

- Input is a complete directed graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}_+$.
- Edge costs are not necessarily symmetric. That is $c(u, v)$ can be different from $c(v, u)$
- Edge costs satisfy assymetric triangle inequality: $c(u, w) \leq c(u, v) + c(v, w)$ for all $u, v, w \in V$.

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Alternate interpretation: given directed graph $G = (V, E)$ find a closed walk that visits all vertices (can visit a vertex more than once).

ATSP

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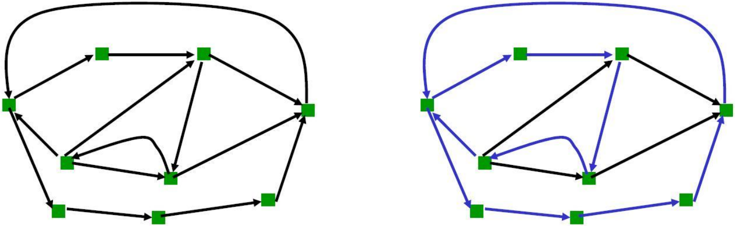


Same as finding a minimum cost connected Eulerian subgraph of G.

Approximation for ATSP

Harder than Metric-TSP

- Simple $\log_2 n$ approximation from 1980.
- Improved to $O(\log n/ \log\log n)$-approximation in 2010.
- Further improved to $O((\log\log n)^c)$-approximation in 2015.

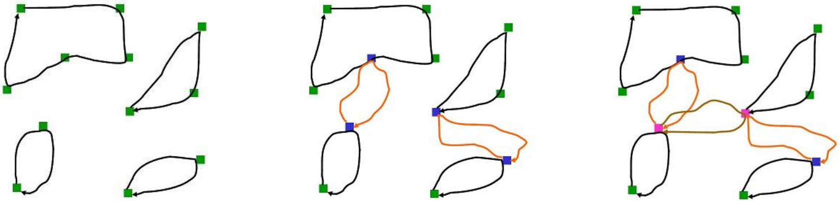
Believed that a constant factor approximation exists via a natural LP relaxation.

The O(log n) Approximation

Recall that a cycle cover is a collection of node disjoint cycles that contain all nodes.

CycleShrinkingAlgorithm($G(V, A), c : A \rightarrow \mathcal{R}^+$):
    If $|V| = 1$ output the trivial cycle consisting of V
    Find a *minimum cost cycle cover* with cycles $C_1, \ldots, C_k$
    From each C_i pick an arbitrary proxy node v_i
    Let $S = \{v_1, v_2, \ldots, v_k\}$
    Recursively solve problem on $G[S]$ to obtain a solution C
    $C' = C \cup C_1 \cup C_2 \ldots C_k$ is a Eulerian graph.
    Shortcut C' to obtain a cycle on V and output C'.

Illustration



Analysis

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Cost of a cycle cover is at most OPT.

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Lemma

The number of vertices shrinks by half in each iteration and hence total of at most $\lceil \log n \rceil$ *cycle covers.*

Hence total cost of all cycle covers is at most $\lceil \log n \rceil \cdot OPT$.