More Approximation Algorithms

Lecture 25

Nov 30, 2016

Formal definition of approximation algorithm

An algorithm $\mathcal{A}$ for an optimization problem X is an α-approximation algorithm if the following conditions hold:

- for each instance I of X the algorithm $\mathcal{A}$ correctly outputs a valid solution to I
- $\mathcal{A}$ is a polynomial-time algorithm
- Letting $OPT(I)$ and $\mathcal{A}(I)$ denote the values of an optimum solution and the solution output by $\mathcal{A}$ on instances I, $OPT(I)/\mathcal{A}(I) \leq \alpha$ *and* $\mathcal{A}(I)/OPT(I) \leq \alpha$. Alternatively:
  - If X is a minimization problem: $\mathcal{A}(I)/OPT(I) \leq \alpha$
  - If X is a maximization problem: $OPT(I)/\mathcal{A}(I) \leq \alpha$

Definition ensures that $\alpha \geq 1$

To be formal we need to say $\alpha(n)$ where $n = |I|$ since in some cases the *approximation ratio* depends on the size of the instance.

Formal definition of approximation algorithm

Unfortunately notation is not consistently used. Some times people use the following convention:

- If X is a minimization problem then $\mathcal{A}(I)/OPT(I) \leq \alpha$ and here $\alpha \geq 1$.
- If X is a maximization problem then $\mathcal{A}(I)/OPT(I) \geq \alpha$ and here $\alpha \leq 1$.

Usually clear from the context.

Part I

Approximation for Load Balancing

Load Balancing

Given n jobs $J_1, J_2, \ldots, J_n$ with sizes $s_1, s_2, \ldots, s_n$ and m identical machines $M_1, \ldots, M_m$ assign jobs to machines to minimize maximum load (also called makespan).

Problem sometimes referred to as multiprocessor scheduling.
Example: 3 machines and 8 jobs with sizes $4, 3, 1, 2, 5, 6, 9, 7$.

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Formally, an assignment is a mapping $f : \{1, 2, \ldots, n\} \rightarrow \{1, \ldots, m\}$.

- The load $\ell_f(j)$ of machine M_j under f is $\sum_{i:f(i)=j} s_i$
- Goal is to find f to minimize $\max_j \ell_f(j)$.

Greedy List Scheduling

List-Scheduling

```
Let J_1, J_2, ..., J_n be an ordering of jobs
for i = 1 to n do
    Schedule job J_i on the currently least loaded machine
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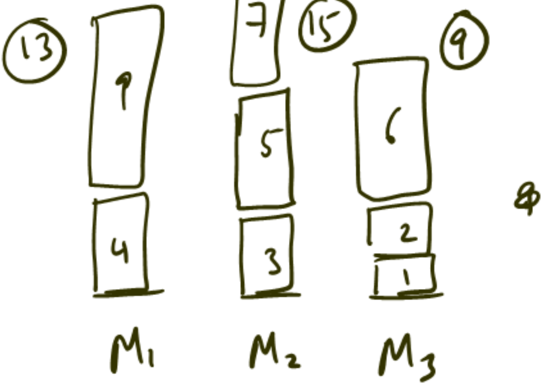

Example

Example: $\mathbf{3}$ machines and $\mathbf{8}$ jobs with sizes $\mathbf{4, 3, 1, 2, 5, 6, 9, 7}$.
Different list: $\mathbf{9, 7, 6, 5, 4, 3, 2, 1}$

Two lower bounds on OPT

OPT is the optimum load

- average load: $OPT \geq \sum_{i=1}^{n} s_i/m$. Why?
- maximum job size: $OPT \geq \max_{i=1}^{n} s_i$. Why?

Analysis of Greedy List Scheduling

Theorem

Let L be makespan of Greedy List Scheduling on a given instance. Then $L \leq 2(1 - 1/m)OPT$ where OPT is the optimum makespan for that instance.

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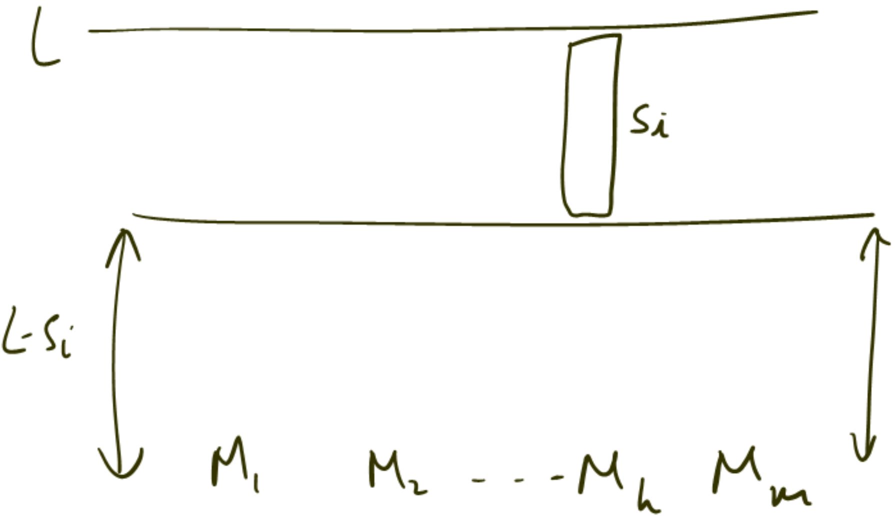
- Let M_h be the machine which achieves the load L for Greedy List Scheduling.
- Let J_i be the job that was last scheduled on M_h.
- Why was J_i scheduled on M_h? It means that M_h was the least loaded machine when J_i was considered. Implies all machines had load at least $L - s_i$ at that time.

$$L - s_i \le \sum_{l=1}^{n} s_l \,/\, m \le OPT$$

$$s_i \le OPT$$



Analysis continued

Lemma

$L - s_i \leq (\sum_{\ell=1}^{i-1} s_\ell)/m$.

Proof.

Since all machines had load at least $L - s_i$ it means that $m(L - s_i) \leq \sum_{\ell=1}^{i-1} s_\ell$ and hence

$$L - s_i \leq (\sum_{\ell=1}^{i-1} s_\ell)/m.$$

□

Analysis continued

But then

$$
\begin{aligned}
L &\leq \Big(\sum_{\ell=1}^{i-1} s_\ell\Big)/m + s_i \\
&\leq \Big(\sum_{\ell=1}^{n} s_\ell\Big)/m + \Big(1-\frac{1}{m}\Big)s_i \\
&\leq OPT + \Big(1-\frac{1}{m}\Big)OPT \\
&\leq \cancel{2\Big(1-\frac{1}{m}\Big)}OPT.
\end{aligned}
$$

$$\Big(2-\frac{1}{m}\Big)\,OPT$$

A Tight Example

Question: Is the analysis of the algorithm tight? That is, are there instances where L is $\cancel{2(1 - 1/m)}OPT$?

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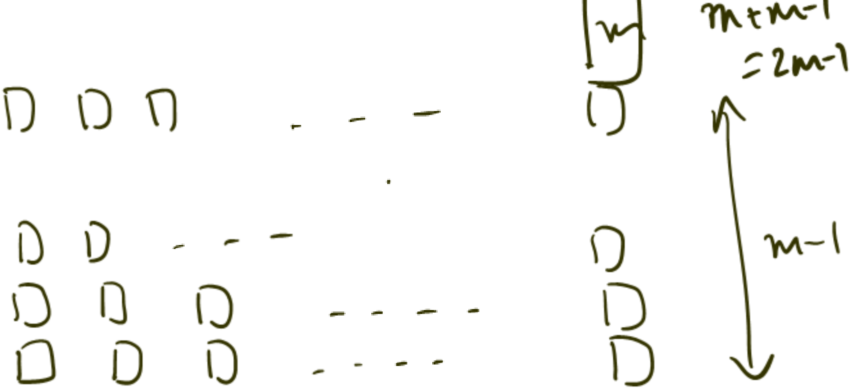

A Tight Example

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Example: $m(m - 1)$ jobs of size 1 and one big job of size m where m is number of machines.

- $OPT = m$. Why?
- If the list has large job at end the schedule created by Greedy is $m + m - 1 = 2m - 1$.

Ordering jobs from largest to smallest

Obvious heuristic: Order jobs in decreasing size order and then use Greedy.

Does it lead to an improved performance in the worst case? How much?

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Theorem

Greedy List Scheduling with jobs sorted from largest to smallest gives a $4/3$-approximation and this is essentially tight.

Analysis

Not so obvious.

If we only use average load and maximum job size as lower bounds on OPT then we cannot improve the bound of 2

Example: $m + 1$ jobs of size 1

- $OPT = 2$
- average load is $1 + 1/m$ and max job size is 1

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Need another lower bound

Another useful lower bound

Lemma

Suppose jobs are sorted, that is $s_1 \geq s_2 \geq \ldots \geq s_n$ *and* $n > m$ *then* $OPT \geq s_m + s_{m+1} \geq 2s_{m+1}$.

Another useful lower bound

Lemma

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Proof.

Consider the first $m+1$ jobs $J_1, \ldots, J_{m+1}$. By pigeon hole principle two of these jobs on same machine. Load on that machine is at least the sum of the smallest two job sizes in the first $m+1$ jobs. □

Proving a $3/2$ bound

Using the new lower bound we will prove a weaker upper bound of $3/2$ rather than the right bound of $4/3$.

As before let M_j be the machine achieving the makespan L and let J_i be the last job assigned to M_j. we have $L - s_i \leq \frac{1}{m}\sum_{\ell=1}^{i-1} s_\ell$. Now a more careful analysis.

- Case 1: If s_i is only job on M_j then $L \leq s_i \leq OPT$.
- Case 2: At least one more job on M_j before s_i.
  - We have seen that $L - s_i \leq OPT$.
  - **Claim:** $s_i \leq OPT/2$
  - Together, we have $L \leq OPT + s_i \leq 3OPT/2$.

Proof of Claim

Since M_j had a job before s_i we have $i > m$.

Hence $s_i \leq s_{m+1}$ becase jobs were sorted. Since $OPT \geq 2s_{m+1}$, we have $s_i \leq s_{m+1} \leq OPT/2$.

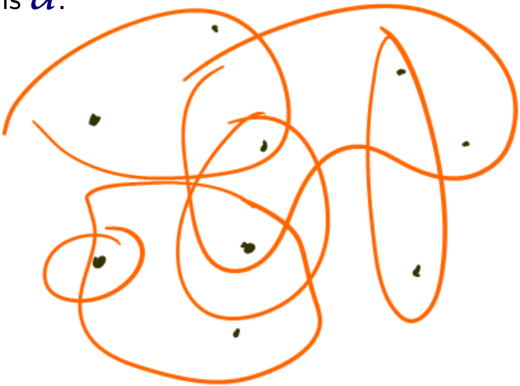
Part II

Approximation for Set Cover

Set Cover

Input: Universe $\mathcal{U}$ of n elements and m subsets $S_1, S_2, \ldots, S_m$ such that $\cup_i S_i = \mathcal{U}$.

Goal: Pick fewest number of subsets to cover all of $\mathcal{U}$ (equivalently, whose union is $\mathcal{U}$.



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Greedy$(\mathcal{U}, S_1, S_2, \ldots, S_m)$

```
Uncovered = U
While Uncovered ≠ ∅ do
    Pick set S_j that covers max number of uncovered elements
    Add S_j to solution
    Uncovered = Uncovered − S_j
endWhile
Output chosen sets
```

Analysis of Greedy

- Let k^* be minimum number of sets to cover $\mathcal{U}$. Let k be number of sets chosen by Greedy.
- Let α_i be number of new elements covered in iteration i.
- Let β_i be number of elements uncovered at end of iteration i. $\beta_0 = n$.

β_0 α_1 α_2

$\underline{n}$ $\beta_1 = n - \alpha_1$ $\beta_2 = \beta_1 - \alpha_2$

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- Let α_i be number of new elements covered in iteration i.
- Let β_i be number of elements uncovered at end of iteration i. $\beta_0 = n$.

Lemma

$\alpha_i \geq \beta_{i-1}/k^*$.

Proof.

Let $\mathcal{U}_i$ be uncovered elements at start of iteration i. All these elements can be covered by some k^* sets since all of $\mathcal{U}$ can be covered by k^* sets. There exists one of those sets that covers at least $\mathcal{U}_i/k^*$ elements. Greedy picks the best set and hence covers at least that many elements. Note $\mathcal{U}_i = \beta_{i-1}$. □

Analysis of Greedy contd

Lemma

$\alpha_i \geq \beta_{i-1}/k^*$.

$$\beta_i = \beta_{i-1} - \alpha_i \leq \beta_{i-1} - \beta_{i-1}/k^* = (1 - 1/k^*)\beta_{i-1}.$$

Hence by induction,

$$\beta_i \leq \beta_0(1 - 1/k^*)^i = n(1 - 1/k^*)^i.$$

Thus, after $k = k^* \ln n$ iterations number number of uncovered elements is at most

$$n(1 - 1/k^*)^{k^* \ln n} \leq ne^{-\ln n} \leq 1.$$

Thus algorithm terminates in at most $k^* \ln n + 1$ iterations. Total number of sets chosen is number of iterations.

Analysis contd

Theorem

Greedy gives a $(\ln n + 1)$*-approximation for Set Cover.*

- Algorithm generalizes to weighted case easily. Pick sets in each iteration based on ratio of elements covered divided by weight. Analysis a bit harder but also gives a $(\ln n + 1)$-approximation.
- Can show a tighter bound of $(\ln d + 1)$ where d is maximum set size.

Analysis contd

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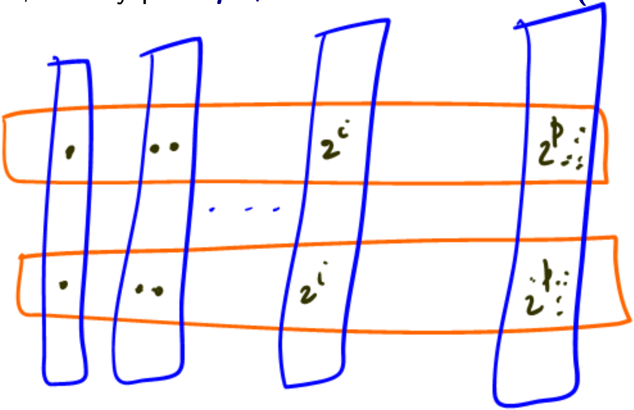
Theorem

Unless $P = NP$ *no* $(\ln n + \epsilon)$*-approximation for Set Cover.*

A bad example for Greedy

$n = 2(1 + 2 + 2^2 + 2^p) = 2(2^{p+1} - 1)$, $m = 2 + 2(p + 1)$, $OPT = 2$, Greedy picks $p + 1$ and hence ratio is $\Omega(\ln n)$.



Advantage of Greedy

Greedy is a simple algorithm. In several scenarios the set system is *implicit* and exponentially large in n. Nevertheless, the Greedy algorithm can be implemented efficiently if there is an oracle that each step picks the best set efficiently.

Max k-Cover

Input: Universe $\mathcal{U}$ of n elements and m subsets $S_1, S_2, \ldots, S_m$ and integer k.

Goal: Pick k subsets to *maximize* number of covered elements.

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    Uncovered = U
    for i = 1 to k do
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    endWhile
    Output chosen k sets
```

Analysis

Similar to previous analysis.

- Let OPT be max number of covered elements to cover $\mathcal{U}$.
- Let α_i be number of new elements covered in iteration i.
- Let γ_i be number of elements covered by greedy after i iterations.
- Let $\beta_i = OPT - \gamma_i$. Define $\beta_0 = OPT$.

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Similar to previous analysis.

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Analysis contd

Lemma

$\alpha_i \geq \beta_{i-1}/k^*$.

$$\beta_i = \beta_{i-1} - \alpha_i \leq \beta_{i-1} - \beta_{i-1}/k = (1 - 1/k)\beta_{i-1}.$$

Hence by induction,

$$\beta_i \leq \beta_0(1 - 1/k)^i = OPT(1 - 1/k)^i.$$

Thus, after k iterations,

$$\beta_k \leq OPT(1 - 1/k)^k \leq OPT/e.$$

Thus $\gamma_k = OPT - \beta_k \geq (1 - 1/e)OPT$.

Analysis contd

Theorem

Greedy gives a $(1 - 1/e)$*-approximation for Max* k*-Coverage.*

Above theorem generalizes to submodular function maximization and has *many* applications.

Theorem (Feige 1998)

Unless $P = NP$ *there is no* $(1 - 1/e - \epsilon)$*-approximation for Max* k*-Coverage for any fixed* $\epsilon > 0$*.*