CS 473: Algorithms

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Fall 2016

Inequalities & QuickSort w.h.p.

Lecture 8

September 16, 2016

Outline

Slick Analysis of Randomized **QuickSort**

Concentration of Mass Around Mean

Markov's Inequality

Chebyshev's Inequality

Chernoff Bound

Randomized **QuickSort**: High Probability Analysis

Part I

Slick analysis of QuickSort

Recall: Randomized **QuickSort**

Randomized **QuickSort**

1. Pick a pivot element *uniformly at random* from the array.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

A Slick Analysis of **QuickSort**

Let $\mathbf{Q(A)}$ be number of comparisons done on input array $\mathbf{A}$:

1. For $1 \leq i < j < n$ let R_{ij} be the event that rank i element is compared with rank j element.
2. X_{ij} is the indicator random variable for R_{ij}. That is, $X_{ij} = 1$ if rank i is compared with rank j element, otherwise 0.

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$$Q(A) = \sum_{1 \leq i < j \leq n} X_{ij}$$

and hence by linearity of expectation,

$$E\left[Q(A)\right] = \sum_{1 \leq i < j \leq n} E\left[X_{ij}\right] = \sum_{1 \leq i < j \leq n} \Pr\left[R_{ij}\right].$$

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$\mathbf{R_{ij}}$ = rank $\mathbf{i}$ element is compared with rank $\mathbf{j}$ element.

Question: What is $\mathbf{Pr[R_{ij}]}$?

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With ranks:	6	4	8	1	2	3	7	5

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With ranks: 6 4 8 1 2 3 7 5

As such, probability of comparing **5** to **8** is $\mathbf{Pr[R_{4,7}]}$.

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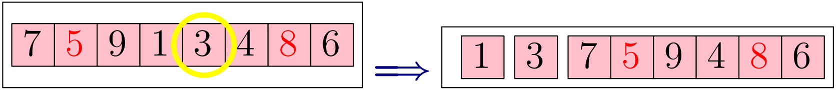

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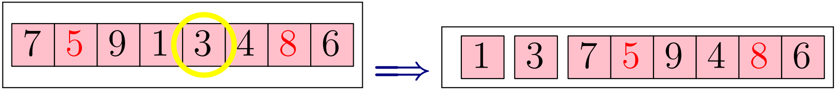

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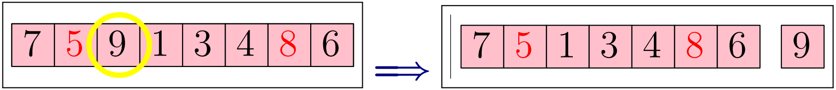
2. If pivot too large (say **9** [rank 8]):



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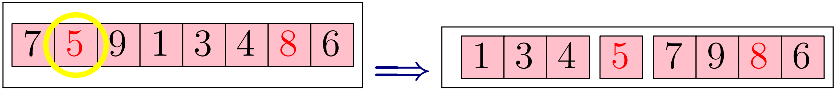

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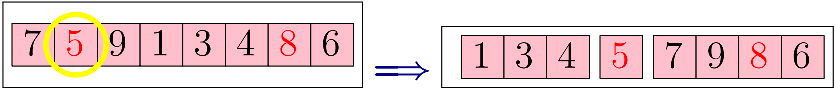
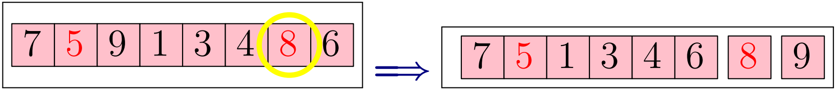


2. If pivot is **8** (rank 7). Bingo!



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Question: What is **Pr[$R_{i,j}$]**?

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2. If pivot is **8** (rank 7). Bingo!

   [7 | 5 | 9 | 1 | 3 | 4 | 8 | 6] ⟹ [7 | 5 | 1 | 3 | 4 | 6] [8] [9]

3. If pivot in between the two numbers (say **6** [rank 5]):

   [7 | 5 | 9 | 1 | 3 | 4 | 8 | 6] ⟹ [5 | 1 | 3 | 4] [6] [7 | 8 | 9]

   5 and **8** will never be compared to each other.

A Slick Analysis of **QuickSort**

Question: What is $\mathbf{Pr[R_{i,j}]}$?

Conclusion:

$\mathbf{R_{i,j}}$ happens if and only if:

> **i**th or **j**th ranked element is the first pivot out of
> **i**th to **j**th ranked elements.

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Proof.

Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be elements of $\mathbf{A}$ in sorted order. Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$

Observation: If pivot is chosen outside $\mathbf{S}$ then all of $\mathbf{S}$ either in left array or right array.

Observation: a_i and a_j separated when a pivot is chosen from $\mathbf{S}$ for the first time. Once separated no comparison.

Observation: a_i is compared with a_j if and only if the first chosen pivot from $\mathbf{S}$ is either a_i or a_j. □

A Slick Analysis of **QuickSort**

Continued...

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Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be sort of $\mathbf{A}$. Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$

Observation: a_i is compared with a_j if and only if the first chosen pivot from S is either a_i or a_j.

Observation: Given that pivot is chosen from S the probability that it is a_i or a_j is exactly $2/|S| = 2/(j - i + 1)$ since the pivot is chosen uniformly at random from the array. □

A Slick Analysis of **QuickSort**

Continued...

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Proof.

Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be sort of A. Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$. Event E when first pivot from S is chosen.

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A Slick Analysis of **QuickSort**

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Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be sort of $\mathbf{A}$. Let $\mathbf{S} = \{a_i, a_{i+1}, \ldots, a_j\}$. Event $\mathbf{E}$ when first pivot from $\mathbf{S}$ is chosen.

Observation: Given $\mathbf{E}$ probability that the pivot is a_i or a_j is exactly $2/|\mathbf{S}| = 2/(j - i + 1)$, *i.e.*

$\mathbf{Pr}[\mathbf{R_{ij}}|\mathbf{E}] = 2/(j - i + 1)$.

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Observation: Given E probability that the pivot is a_i or a_j is exactly $2/|S| = 2/(j-i+1)$, *i.e.*

$\Pr[R_{ij}|E] = 2/(j-i+1)$.

Since $\Pr[E] = 1$, we get $\Pr[R_{ij}] = 2/(j-i+1)$.

□

A Slick Analysis of **QuickSort**

Continued...

So far we know

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A Slick Analysis of **QuickSort**

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A Slick Analysis of **QuickSort**

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$\mathbf{Pr}[R_{ij}] = \frac{2}{j-i+1}$.

$$\mathbf{E}\Big[Q(A)\Big] = 2\sum_{i=1}^{n-1}\sum_{i<j}^{n}\frac{1}{j-i+1}$$

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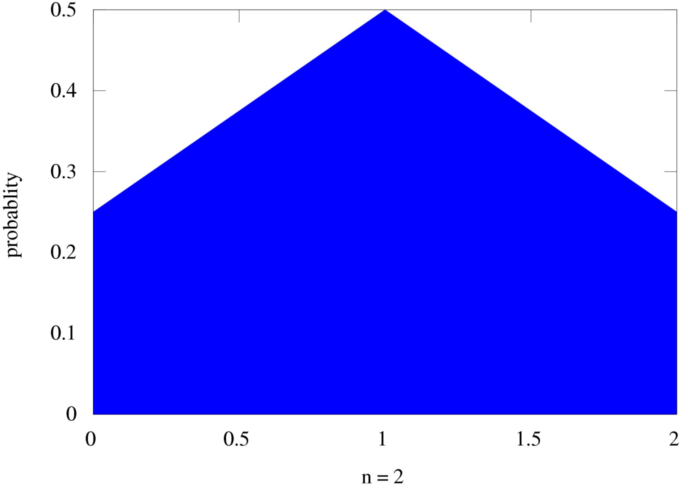
$$\leq 2nH_n = O(n\log n)$$

Part II

Inequalities

Massive randomness.. Is not that random.

Consider flipping a fair coin $\mathbf{n}$ times independently, head gives $\mathbf{1}$, tail gives zero. How many $\mathbf{1}$s? Binomial distribution: $\mathbf{k}$ w.p. $\binom{n}{k} 1/2^n$.



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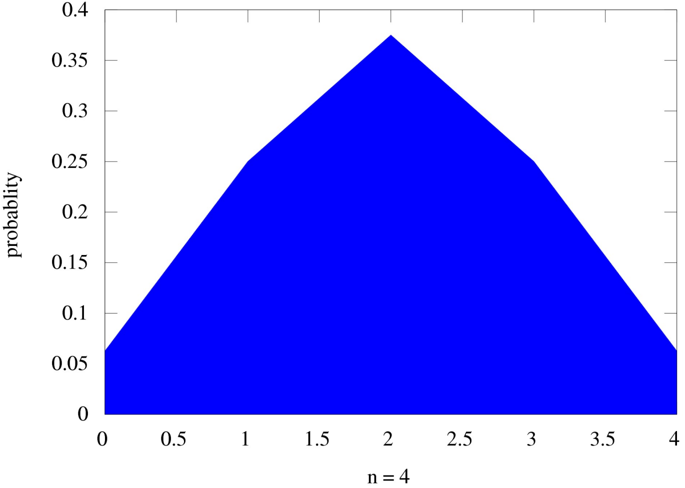

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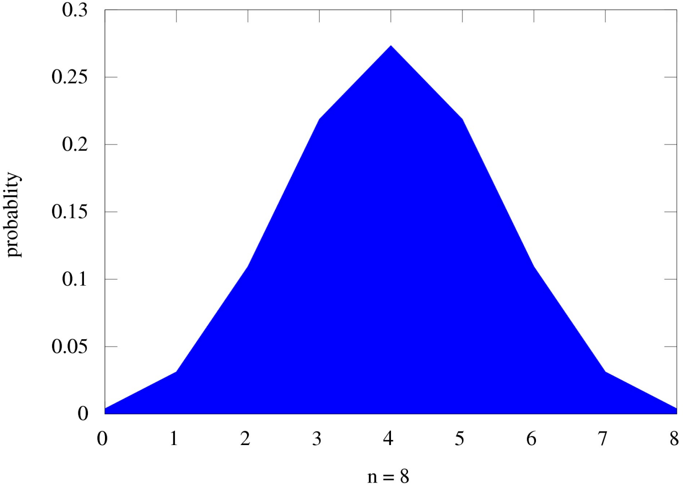

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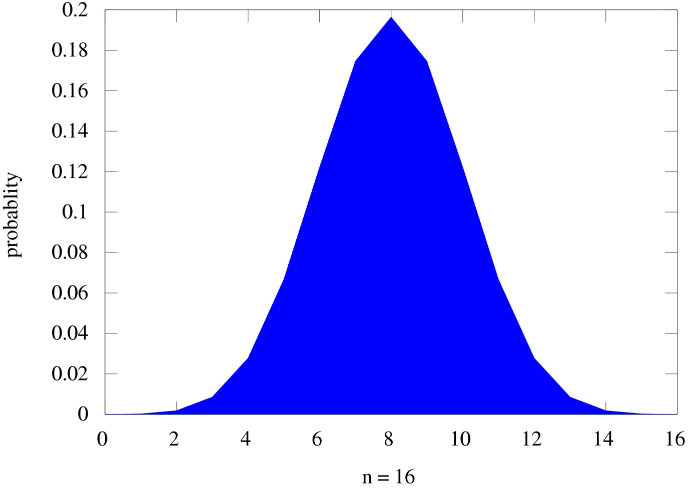

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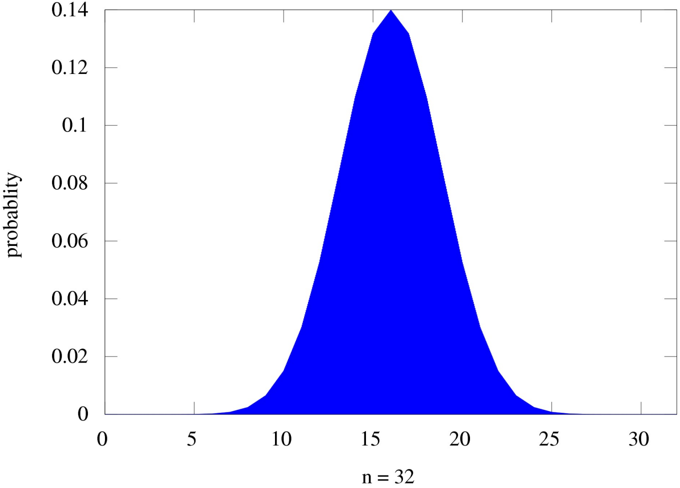

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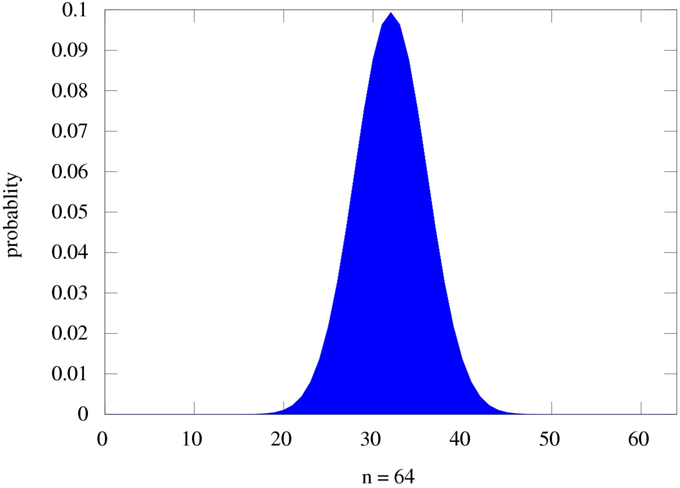

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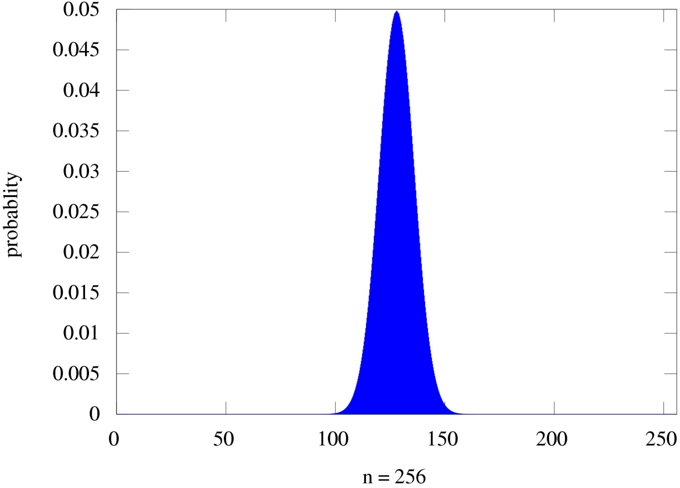

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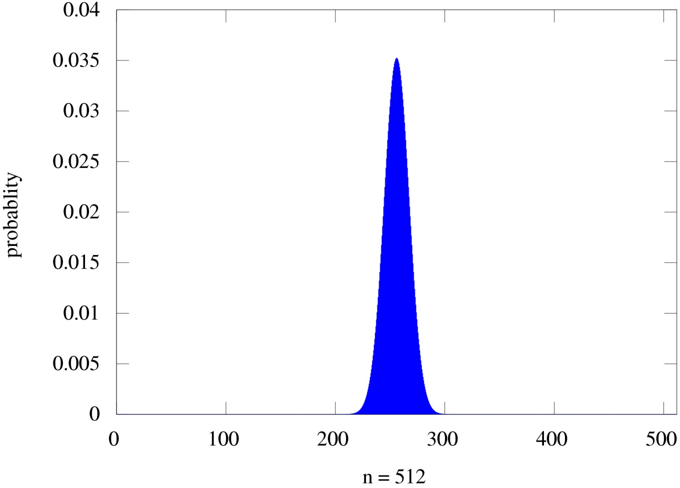

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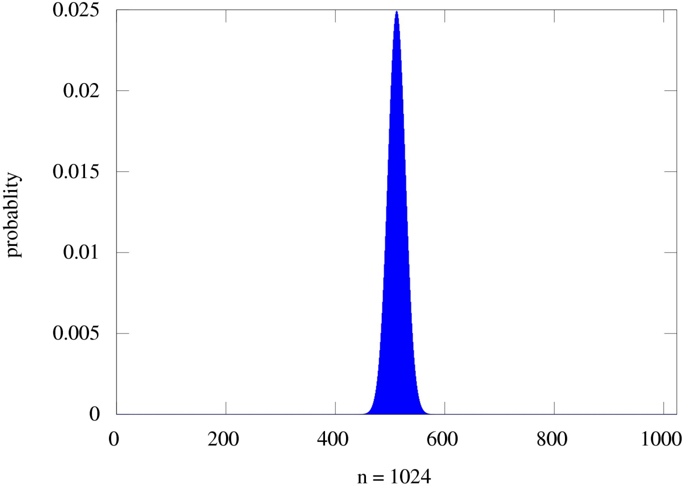

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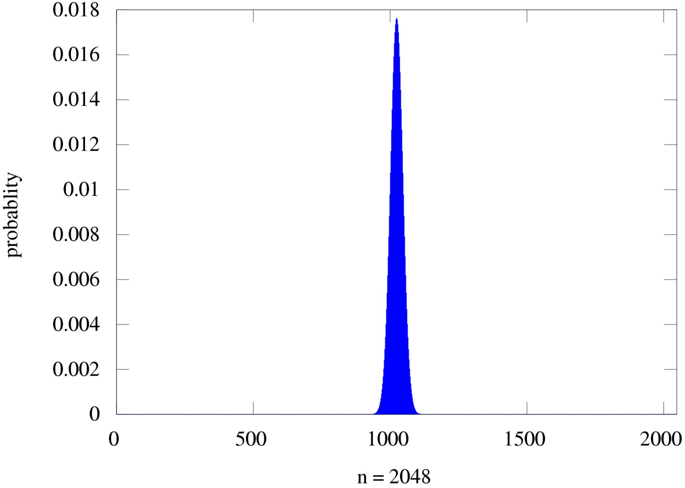

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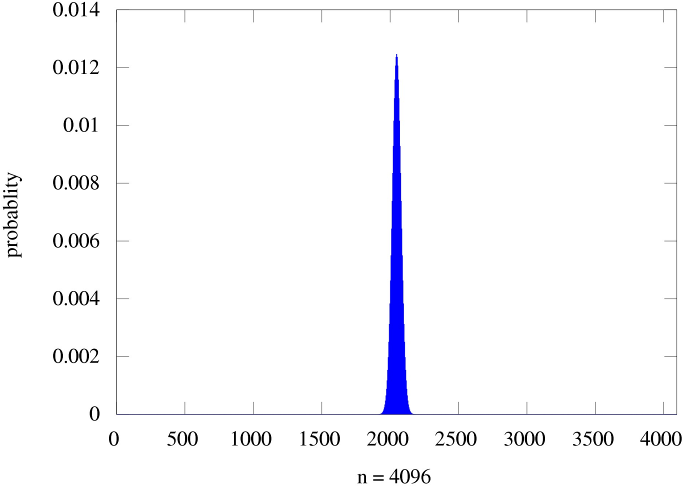

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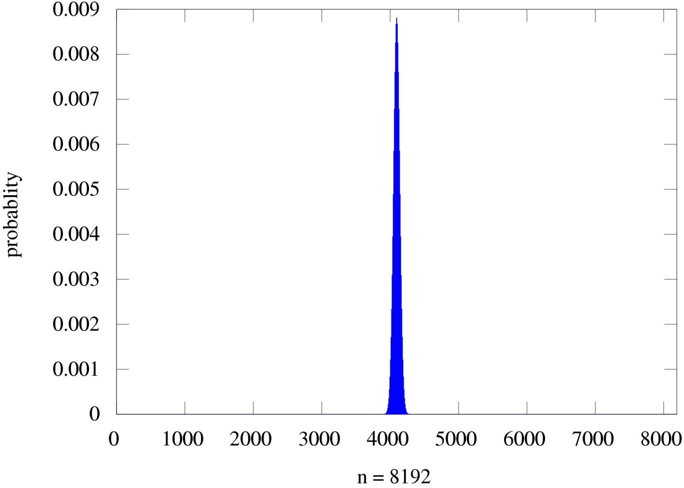

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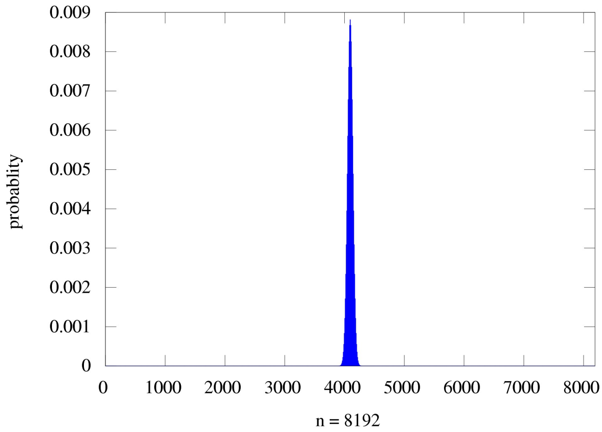


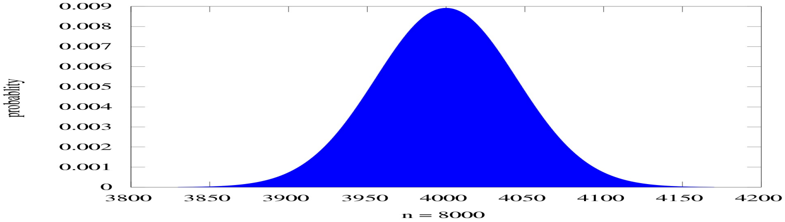
This is known as **concentration of mass**.
This is a very special case of the **law of large numbers**.

Side note...

Law of large numbers (weakest form)...

Informal statement of law of large numbers

For **n** large enough, the middle portion of the binomial distribution looks like (converges to) the normal/Gaussian distribution.



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Intuitive conclusion

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.

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Use of well known inequalities in analysis.

Randomized **QuickSort**: A possible analysis

Analysis

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- Suppose $\mathbf{Pr[Q \geq 10n lg n] \leq c}$. Also we know that $\mathbf{Q \leq n^2}$.
- $\mathbf{E[Q] \leq 10n \log n + (n^2 - 10n \log n)c}$.

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- $\mathbf{E[Q] \leq 10n \log n + (n^2 - 10n \log n)c}$.

Question:

How to find $\mathbf{c}$, or in other words bound $\mathbf{Pr[Q \geq 10n \log n]}$?

Markov's Inequality

Markov's inequality

Let X be a **non-negative** random variable over a probability space $(\Omega, \Pr)$. For any $a > 0$,

$$\Pr[X \geq a] \leq \frac{E[X]}{a}$$

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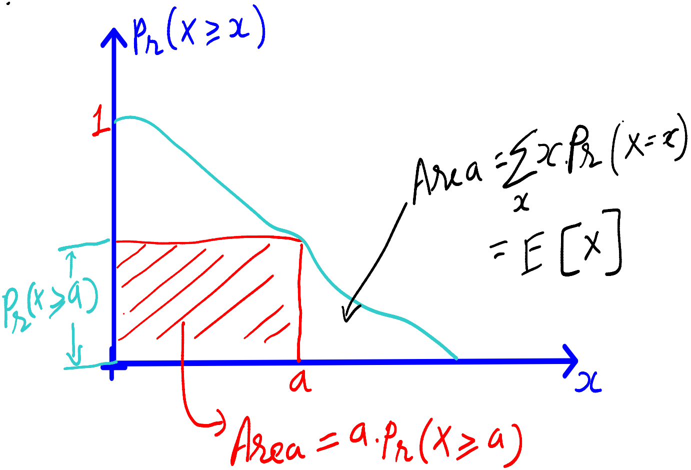
Proof:

$$\begin{aligned} E[X] &= \sum_{\omega \in \Omega} X(\omega) \Pr[\omega] \\ &\geq \sum_{\omega \in \Omega,\ X(\omega) \geq a} X(\omega) \Pr[\omega] \\ &\geq a \sum_{\omega \in \Omega,\ X(\omega) \geq a} \Pr[\omega] \\ &= a \Pr[X \geq a] \end{aligned}$$

Markov's Inequality: Proof by Picture



Example: Balls in a bin

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Question

How large $\mathbf{k}$ needs to be before our estimated value $\mathbf{p}$ is close to $\mathbf{p}^*$?

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A rough estimate through Markov's inequality.

Lemma

For any $\mathbf{k} \geq \mathbf{1}$, $\mathbf{Pr[p \geq 2p^*] \leq \frac{1}{2}}$

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Proof.

- For each $\mathbf{1} \leq \mathbf{i} \leq \mathbf{k}$ define random variable $\mathbf{X_i}$, which is $\mathbf{1}$ if $\mathbf{i}^{\mathbf{th}}$ ball is black, otherwise $\mathbf{0}$.
- $\mathbf{E}[\mathbf{X_i}] = \mathbf{Pr}[\mathbf{X_i} = \mathbf{1}] = \mathbf{p}^*$.

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Proof.

- For each $\mathbf{1} \leq \mathbf{i} \leq \mathbf{k}$ define random variable $\mathbf{X_i}$, which is $\mathbf{1}$ if $\mathbf{i^{th}}$ ball is black, otherwise $\mathbf{0}$.
- $\mathbf{E[X_i] = Pr[X_i = 1] = p^*}$.
- $\mathbf{B = \sum_{i=1}^{k} X_i}$, then $\mathbf{E[B] = \sum_{i=1}^{k} E[X_i] = kp^*}$. $\mathbf{p = B/k}$.

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A rough estimate through Markov's inequality.

Lemma

For any $\mathbf{k \geq 1}$, $\mathbf{Pr[p \geq 2p^*] \leq \frac{1}{2}}$

Proof.

- For each $\mathbf{1 \leq i \leq k}$ define random variable $\mathbf{X_i}$, which is $\mathbf{1}$ if $\mathbf{i^{th}}$ ball is black, otherwise $\mathbf{0}$.
- $\mathbf{E[X_i] = Pr[X_i = 1] = p^*}$.
- $\mathbf{B = \sum_{i=1}^{k} X_i}$, then $\mathbf{E[B] = \sum_{i=1}^{k} E[X_i] = kp^*}$. $\mathbf{p = {}^{B}/_{k}}$.
- Markov's inequality gives, $\mathbf{Pr[p \geq 2p^*] =}$

$$\mathbf{Pr\left[\frac{B}{k} \geq 2p^*\right] = Pr[B \geq 2kp^*] = Pr[B \geq 2\,E[B]] \leq \frac{1}{2}}$$

Chebyshev's Inequality: Variance

Variance

Given a random variable $\mathbf{X}$ over probability space $(\boldsymbol{\Omega}, \mathbf{Pr})$, variance of $\mathbf{X}$ is the measure of how much does it deviate from its mean value. Formally, $\mathbf{Var(X) = E\left[(X - E[X])^2\right] = E\left[X^2\right] - E[X]^2}$

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Intuitive Derivation

Define $\mathbf{Y} = (\mathbf{X} - \mathbf{E[X]})^2 = \mathbf{X}^2 - 2\mathbf{X}\,\mathbf{E[X]} + \mathbf{E[X]}^2$.

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$$\begin{aligned}
\mathbf{Var}(\mathbf{X}) &= \mathbf{E}[\mathbf{Y}] \\
&= \mathbf{E}\left[\mathbf{X}^2\right] - 2\,\mathbf{E}[\mathbf{X}]\,\mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{X}]^2 \\
&= \mathbf{E}\left[\mathbf{X}^2\right] - \mathbf{E}[\mathbf{X}]^2
\end{aligned}$$

Chebyshev's Inequality: Variance

Independence

Random variables $\mathbf{X}$ and $\mathbf{Y}$ are called mutually independent if

$$\forall x, y \in \mathbb{R}, \; \Pr[X = x \wedge Y = y] = \Pr[X = x] \Pr[Y = y]$$

Lemma

If $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent random variables then* $\mathbf{Var}(X + Y) = \mathbf{Var}(X) + \mathbf{Var}(Y)$.

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Chebyshev's Inequality

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Given $\mathbf{a} \geq \mathbf{0}$, $\mathbf{Pr}[|\mathbf{X} - \mathbf{E}[\mathbf{X}]| \geq \mathbf{a}] \leq \frac{\mathbf{Var}(\mathbf{X})}{\mathbf{a}^2}$

Chebyshev's Inequality

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Given $a \geq 0$, $\Pr[|X - E[X]| \geq a] \leq \frac{Var(X)}{a^2}$

Proof.

$Y = (X - E[X])^2$ is a non-negative random variable. Apply Markov's Inequality to Y for a^2.

$$\Pr\left[Y \geq a^2\right] \leq E[Y]/a^2 \Leftrightarrow \Pr\left[(X - E[X])^2 \geq a^2\right] \leq Var(X)/a^2$$
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□

$\Pr[X \leq E[X] - a] \leq Var(X)/a^2$ AND $\Pr[X \geq E[X] + a] \leq Var(X)/a^2$

Example: Balls in a bin (contd)

Lemma

For $\mathbf{0} < \epsilon < \mathbf{1}$ *and* $\mathbf{k} \geq \mathbf{1}$, $\mathbf{Pr}[|\mathbf{p} - \mathbf{p}^*| > \epsilon] \leq {}^{1}/_{\mathbf{k}\epsilon^2}$.

Proof.

- Recall: $\mathbf{X_i}$ is $\mathbf{1}$ is $\mathbf{i^{th}}$ ball is black, else $\mathbf{0}$, $\mathbf{B} = \sum_{\mathbf{i=1}}^{\mathbf{k}} \mathbf{X_i}$. $\mathbf{E[X_i] = p^*}$, $\mathbf{E[B] = kp^*}$. $\mathbf{p} = {}^{\mathbf{B}}/_{\mathbf{k}}$.

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- $\mathbf{Var(X_i) = E\left[X_i^2\right] - E[X_i]^2 = E[X_i] - E[X_i]^2 = p^*(1 - p^*)}$

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- $\mathbf{Var(B)} = \sum_i \mathbf{Var(X_i)} = \mathbf{kp^*(1 - p^*)}$ (Exercise)

Example:Balls in a bin (contd)

Lemma

For $0 < \epsilon < 1$ *and* $k \geq 1$, $\Pr[|p - p^*| > \epsilon] \leq 1/k\epsilon^2$.

Proof.

- Recall: X_i is 1 is i^{th} ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = B/k$.
- $Var(X_i) = E\left[X_i^2\right] - E[X_i]^2 = E[X_i] - E[X_i]^2 = p^*(1 - p^*)$
- $Var(B) = \sum_i Var(X_i) = kp^*(1 - p^*)$ (Exercise)
- $$\begin{aligned} \Pr[|B/k - p^*| \geq \epsilon] &= \Pr[|B - kp^*| \geq k\epsilon] \\ \text{(Chebyshev)} &\leq Var(B)/k^2\epsilon^2 = kp^*(1-p^*)/k^2\epsilon^2 \\ &< 1/k\epsilon^2 \end{aligned}$$

□

Chernoff Bound

Lemma

Let $X_1, \ldots, X_k$ *be* k *independent random variables such that, for each* $i \in [1, k]$, X_i *equals* 1 *with probability* p_i*, and* 0 *with probability* $(1 - p_i)$.

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$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{\frac{-\delta^2\mu}{3}}$$

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$$\mathbf{Pr}[|\mathbf{X} - \mu| \geq \delta\mu] \leq 2\mathbf{e}^{\frac{-\delta^2\mu}{3}}$$

$\mathbf{Pr}[\mathbf{X} \geq (1 + \delta)\mu] \leq \mathbf{e}^{\frac{-\delta^2\mu}{3}}$ *and* $\mathbf{Pr}[\mathbf{X} \leq (1 - \delta)\mu] \leq \mathbf{e}^{\frac{-\delta^2\mu}{2}}$

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Proof.

In notes! □

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For any $\mathbf{0} < \epsilon < \mathbf{1}$, *and* $\mathbf{k} \geq \mathbf{1}$, $\mathbf{Pr}[|\mathbf{p} - \mathbf{p}^*| > \epsilon] \leq \mathbf{2e}^{-\frac{\mathbf{k}\epsilon^2}{3}}$.

Proof.

Recall: $\mathbf{X_i}$ is $\mathbf{1}$ is $\mathbf{i^{th}}$ ball is black, else $\mathbf{0}$, $\mathbf{B} = \sum_{\mathbf{i=1}}^{\mathbf{k}} \mathbf{X_i}$.
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&= \Pr[|B - kp^*| \geq k\epsilon] \\
&= \Pr\left[|B - kp^*| \geq \left(\frac{\epsilon}{p^*}\right)kp^*\right] \\
(\text{Chernoff}) &\leq 2e^{-\frac{\epsilon^2}{3p^{*2}}kp^*} = 2e^{-\frac{k\epsilon^2}{3p^*}} \\
(p^* \leq 1) &\leq 2e^{-\frac{k\epsilon^2}{3}}
\end{aligned}
$$

Example Summary

The problem was to estimate the fraction of black balls $\mathbf{p}^*$ in a bin filled with white and black balls. Our estimate was $\mathbf{p} = \frac{\mathbf{B}}{\mathbf{k}}$ instead, where out of $\mathbf{k}$ draws (with replacement) $\mathbf{B}$ balls turns out black.

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Part III

Randomized **QuickSort** (Contd.)

Randomized **QuickSort**: Recall

Input: Array **A** of **n** numbers. **Output:** Numbers in sorted order.

Randomized **QuickSort**

1. Pick a pivot element *uniformly at random* from **A**.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

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Note: On *every* input randomized **QuickSort** takes $\mathbf{O(n \log n)}$ time in expectation. On *every* input it may take $\mathbf{\Omega(n^2)}$ time with some small probability.

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Question: With what probability it takes $\mathbf{O(n \log n)}$ time?

Randomized **QuickSort**: High Probability Analysis

Informal Statement

Random variable $\mathbf{Q(A)} = \#$ comparisons done by the algorithm.

We will show that $\mathbf{Pr[Q(A) \leq 32n \ln n] \geq 1 - 1/n^3}$.

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If $\mathbf{n = 100}$ then this gives $\mathbf{Pr[Q(A) \leq 32n \ln n] \geq 0.99999}$.

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Outline of the proof

- If depth of recursion is $\mathbf{k}$ then $\mathbf{Q(A) \leq kn}$.
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  1. Gocus on a single element. Prove that it "participates" in $> \mathbf{32 \ln n}$ levels with probability at most $\mathbf{1/n^4}$.
  2. By union bound, any of the $\mathbf{n}$ elements participates in $> \mathbf{32 \ln n}$ levels with probability at most

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  2. By union bound, any of the n elements participates in $> 32 \ln n$ levels with probability at most $1/n^3$.

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  2. By union bound, any of the $\mathbf{n}$ elements participates in $> \mathbf{32 \ln n}$ levels with probability at most $\mathbf{1/n^3}$.
  3. Therefore, all elements participate in $\leq \mathbf{32 \ln n}$ w.p. $\mathbf{(1 - 1/n^3)}$.

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Randomized **QuickSort**: High Probability Analysis

- If $\mathbf{k}$ levels of recursion then $\mathbf{kn}$ comparisons.
- Fix an element $\mathbf{s} \in \mathbf{A}$. We will track it at each level.
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- We call $\mathbf{s}$ lucky in $\mathbf{i^{th}}$ iteration, if balanced split: $|\mathbf{S_{i+1}}| \leq \mathbf{(3/4)}|\mathbf{S_i}|$ and $|\mathbf{S_i} \setminus \mathbf{S_{i+1}}| \leq \mathbf{(3/4)}|\mathbf{S_i}|$.

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- If $\rho =$#lucky rounds in first $\mathbf{k}$ rounds, then $|\mathbf{S_k}| \leq \mathbf{(3/4)}^{\rho}\mathbf{n}$.
- For $|\mathbf{S_k}| = \mathbf{1}$, $\rho = \mathbf{4 \ln n} \geq \mathbf{\log_{4/3} n}$ suffices.

How may rounds before $4 \ln n$ lucky rounds?

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$$\begin{aligned} \Pr[\rho \leq 4 \ln n] &= \Pr[\rho \leq k/8] \\ &= \Pr[\rho \leq (1-\delta)\mu] \end{aligned}$$

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- Clearly, $\rho = \sum_{i=1}^{k} X_i$. Let $\mu = E[\rho] = \frac{k}{2}$.
- Set $k = 32 \ln n$ and $\delta = \frac{3}{4}$. $(1 - \delta) = \frac{1}{4}$.

Probability of NOT getting $4 \ln n$ lucky rounds out of $32 \ln n$ rounds is,

$$\begin{aligned} \Pr[\rho \leq 4 \ln n] &= \Pr[\rho \leq k/8] \\ &= \Pr[\rho \leq (1-\delta)\mu] \\ \text{(Chernoff)} \quad &\leq e^{\frac{-\delta^2 \mu}{2}} \\ &= e^{-\frac{9k}{64}} \\ &= e^{-4.5 \ln n} \leq \frac{1}{n^4} \end{aligned}$$

Randomized **QuickSort** w.h.p. Analysis

- n input elements. Probability that depth of recursion in **QuickSort** $> 32 \ln n$ is at most $\frac{1}{n^4} * n = \frac{1}{n^3}$.

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Theorem

With high probability (i.e., $1 - \frac{1}{n^3}$) the depth of the recursion of **QuickSort** *is* $\leq 32 \ln n$*. Due to* n *comparisons in each level, with high probability, the running time of* **QuickSort** *is* $O(n \ln n)$.

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Q: How to increase the probability?