Approximation Algorithms using Linear Programming

Lecture 21 November 10, 2015

Weighted Vertex Cover problem

 $\mathbf{G} = (\mathbf{V}, \mathbf{E}).$

Each vertex $\mathbf{v} \in \mathbf{V}$: cost $\mathbf{c}_{\mathbf{v}}$.

- vertex cover: subset of vertices **V** so each edge is covered.
- NP-Hard
- ...unweighted Vertex Cover problem.
- ... write as an integer program (IP):

- \bigcirc minimize total cost: $\min \sum_{\mathbf{v} \in \mathbf{V}} x_{\mathbf{v}} \mathbf{c}_{\mathbf{v}}$.

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- **∀vu** ∈ **E**: covered. $\implies x_{\mathsf{v}} \lor x_{\mathsf{u}}$ true. $\implies x_{\mathsf{v}} + x_{\mathsf{u}} \ge 1$.
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- MP-Hard.
- 2 relax the integer program.
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- ① $x_{\mathsf{v}} \in \{0,1\}$ replaced by $0 \leq x_{\mathsf{v}} \leq 1$. The resulting LP is

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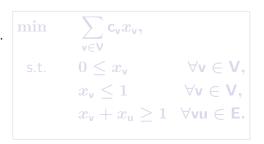
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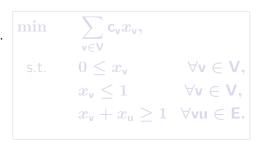
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- **1** Optimal solution to this LP: $\widehat{x_{\mathbf{v}}}$ value of var $X_{\mathbf{v}}$, $\forall \mathbf{v} \in \mathbf{V}$.
- ② optimal value of LP solution is $\widehat{\alpha} = \sum_{\mathbf{v} \in \mathbf{V}} \mathbf{c}_{\mathbf{v}} \widehat{x_{\mathbf{v}}}$.
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- Any valid solution to IP is valid solution for LP!
- $\widehat{\alpha} \leq \alpha^I.$ Integral solution not better than LP.
- o Got fractional solution (i.e., values of $\widehat{x_{v}}$).
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- ② If $\widehat{x_{\mathsf{v}}} = 1$ then include in solution!
- $ext{ @ }$ If $\widehat{x_{\mathsf{v}}} = 0$ then do include in solution.
- ① if $\widehat{x_{\mathbf{v}}} = 0.9 \implies \mathrm{LP}$ considers \mathbf{v} as being $\mathbf{0.9}$ useful.
- 5 The LP puts its money where its belief is...
- $\underline{0}$... $\widehat{\alpha}$ value is a function of this "belief" generated by the LP.
- Big idea: Trust LP values as guidance to usefulness of vertices.
- Pick all vertices threshold of usefulness according to LP.
- lacktriangle Claim: S a valid vertex cover, and cost is low.
- $\widehat{x_{\mathsf{v}}}, \widehat{x_{\mathsf{u}}} \in (0,1) \implies \widehat{x_{\mathsf{v}}} \ge 1/2 \text{ or } \widehat{x_{\mathsf{u}}} \ge 1/2$ $\implies \mathsf{v} \in S \text{ or } \mathsf{u} \in S \text{ (or both)}.$ $\implies S \text{ covers all the edges of } G$

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- lacktriangledown consider vertex $oldsymbol{v}$ and fractional value $\widehat{oldsymbol{x}_{oldsymbol{v}}}.$
- ② If $\widehat{x_{\mathsf{v}}} = 1$ then include in solution!
- **3** If $\widehat{x_{\mathbf{v}}} = 0$ then do not include in solution.
- if $\widehat{x_{\mathbf{v}}} = 0.9 \implies \text{LP}$ considers \mathbf{v} as being 0.9 useful.
- The LP puts its money where its belief is...
- \odot ... $\widehat{\alpha}$ value is a function of this "belief" generated by the LP.
- Big idea: Trust LP values as guidance to usefulness of vertices.
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6

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Cost of solution

Cost of S:

$$\mathsf{c}_S = \sum_{\mathsf{v} \in S} \mathsf{c}_\mathsf{v} = \sum_{\mathsf{v} \in S} 1 \cdot \mathsf{c}_\mathsf{v} \leq \sum_{\mathsf{v} \in S} 2\widehat{x_\mathsf{v}} \cdot \mathsf{c}_\mathsf{v} \leq 2 \sum_{\mathsf{v} \in \mathsf{V}} \widehat{x_\mathsf{v}} \mathsf{c}_\mathsf{v} = 2\widehat{\alpha} \leq 2\alpha^I,$$

since $\widehat{x_{\mathbf{v}}} \geq 1/2$ as $\mathbf{v} \in S$.

 $lpha^I$ is cost of the optimal solution \implies

Theorem

The Weighted Vertex Cover problem can be 2-approximated by solving a single LP. Assuming computing the LP takes polynomial time, the resulting approximation algorithm takes polynomial time.

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Or not - boring, boring, boring.

- Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
- Not aware of any other 2-approximation algorithm does not use LP. (For the weighted case!)
- Solving a relaxation of an optimization problem into a LP provides us with insight.
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21.2: Revisiting Set Cover

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Revisiting **Set Cover**

- Purpose: See new technique for an approximation algorithm.
- ② Not better than greedy algorithm already seen $O(\log n)$ approximation.

Set Cover

```
Instance: (S, \mathcal{F})
```

S: set of n elements

 \mathfrak{F} : family of subsets of S, s.t. $\bigcup_{X \in \mathfrak{F}} X = S$.

Question: The set $\mathcal{X}\subseteq \mathcal{F}$ such that \mathcal{X} contains as few sets

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$$egin{aligned} \min & & lpha & = \sum_{U \in \mathcal{F}} x_U, \ & ext{s.t.} & & x_U \in \{0,1\} & & orall U \in \mathcal{F}, \ & & \sum_{U \in \mathcal{F}, s \in U} x_U \geq 1 & & orall s \in S. \end{aligned}$$

Next, we relax this IP into the following LP.

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- **1** LP solution: $\forall U \in \mathcal{F}$, $\widehat{x_U}$, and $\widehat{\alpha}$.
- ② Opt IP solution: $\forall U \in \mathfrak{F}, \, x_U^I$, and α^I .
- Use LP solution to guide in rounding process
- $ext{ } ext{ }$
- ullet If $\widehat{x_U}$ close to 0 do not.
- **1** Idea: Pick $U \in \mathfrak{F}$: randomly choose U with **probability** $\widehat{x_U}$.
- \odot Resulting family of sets 9.
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Set Cover – Rounding continued

- **9** Sol: Repeat rounding $m=10 \lceil \lg n \rceil = O(\log n)$ times.
- \mathfrak{G}_i : random cover computed in ith iteration.
- ullet $\mathcal{H}=\cup_i \mathcal{G}_i$. Return \mathcal{H} as the required cover.

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Cost of solution

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The result

Theorem

By solving an LP one can get an $O(\log n)$ -approximation to set cover by a randomized algorithm. The algorithm succeeds with high probability.

Same algorithms works for...

Corollary

By solving an LP one can get an $O(\log n)$ -approximation to set cover by a randomized algorithm. The algorithm also works for the weighted case.

$$egin{aligned} \min & & lpha & = \sum_{U \in \mathfrak{F}} w_U x_U \ & & 0 \leq x_U \leq 1 & & orall U \in \mathfrak{F}, \ & & \sum_{U \in \mathfrak{F}, s \in U} x_U \geq 1 & & orall s \in S. \end{aligned}$$

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Rounding algorithm as before...

Cost of solution (weighted case)...

Same same, not the same.

- Fractional LP solution. Target: $\widehat{\alpha} \quad \forall U \in \mathfrak{F}: \widehat{x_U} \in [0,1].$
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- \bullet Have: $\mathbf{E}[\operatorname{cost} \operatorname{of} \mathfrak{S}_i] \leq \alpha^I$.
- \Longrightarrow Each iteration expected cost of cover \leq cost of optimal solution (i.e., α^I).
- Expected cost of the solution is

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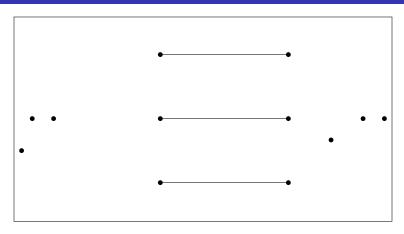
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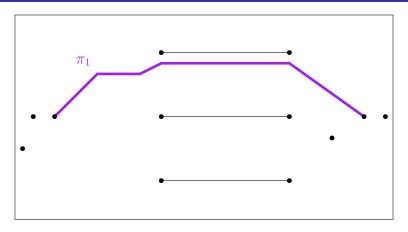
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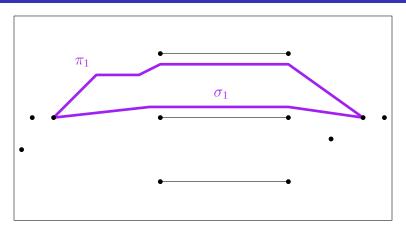
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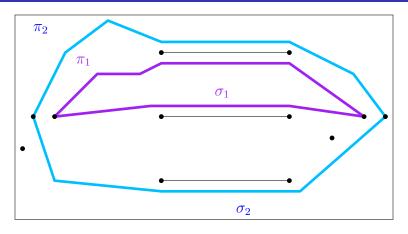
- Fractional LP solution. Target: $\widehat{\alpha} \quad \forall U \in \mathfrak{F}: \widehat{x_U} \in [0,1].$
- **②** Integral opt solution. Target: $lpha^I$. $orall U \in \mathfrak{F}$: $x_U^I \in \{0,1\}$.
- Rounding. ∀U ∈ 𝒯: Pr[X_U = 1] = $\widehat{x_U}$.
 E[cost 𝔞_i] = $\sum_{U ∈ 𝒯}$ E[w_UX_U] = $\sum_{U ∈ 𝒯}$ w_U $\widehat{x_U}$ = $\widehat{\alpha} ≤ \alpha^I$.
- **1** Have: $\mathbf{E}[\operatorname{cost} \operatorname{of} \mathfrak{S}_i] \leq \alpha^I$.
- Each iteration expected cost of cover \leq cost of optimal solution (i.e., α^I).
- Expected cost of the solution is

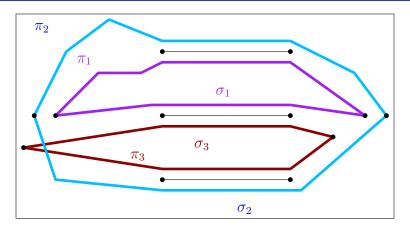
$$\mathsf{c}_{\mathfrak{H}} \leq \sum_{i=1}^{O(\log n)} \mathsf{c}_{B_i} \leq m lpha^I = Oig(lpha^I \log nig).$$

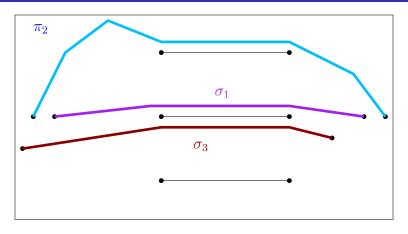


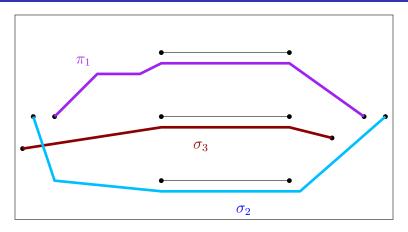












- $oldsymbol{0}$ $oldsymbol{G}$: graph. $oldsymbol{n}$ vertices.
- $oldsymbol{arphi}_i,\, \sigma_i$ paths with the same endpoints $oldsymbol{\mathsf{v}}_i, oldsymbol{\mathsf{u}}_i \in oldsymbol{\mathsf{V}}(oldsymbol{\mathsf{G}})$, for $i=1,\ldots,t$.
- 3 Rule I: Send one unit of flow from \mathbf{v}_i to \mathbf{u}_i .
- **1** Rule II: Choose whether to use π_i or σ_i .
- Target: No edge in G is being used too much.

Definition

Given a set X of paths in a graph G, the **congestion** of X is the maximum number of paths in X that use the same edge.

 \bullet IP \Longrightarrow LP:

min
$$w$$
 s.t. $x_i \geq 0$ $i=1,\ldots,t,$ $x_i \leq 1$ $i=1,\ldots,t,$ $\sum_{\mathbf{e} \in \pi_i} x_i + \sum_{\mathbf{e} \in \sigma_i} (1-x_i) \leq w$ $\forall \mathbf{e} \in E.$

- ② $\widehat{x_i}$: value of x_i in the optimal LP solution.
- $oldsymbol{\widehat{w}}$: value of $oldsymbol{w}$ in LP solution.
- **1** Optimal congestion must be bigger than $\widehat{\boldsymbol{w}}$.
- **5** X_i : random variable one with probability $\hat{x_i}$, and zero otherwise.
- **1** If $X_i = 1$ then use π to route from \mathbf{v}_i to \mathbf{u}_i .
- Otherwise use σ_i .

- **①** Congestion of **e** is $Y_{\mathsf{e}} = \sum_{\mathsf{e} \in \pi_i} X_i + \sum_{\mathsf{e} \in \sigma_i} (1 X_i)$.
- And in expectation

$$egin{aligned} lpha_{ ext{e}} &= ext{E}ig[Y_{ ext{e}}ig] = ext{E}igg[\sum_{ ext{e} \in \pi_i} X_i + \sum_{ ext{e} \in \sigma_i} (1 - X_i)igg] \ &= \sum_{ ext{e} \in \pi_i} ext{E}ig[X_iig] + \sum_{ ext{e} \in \sigma_i} ext{E}ig[(1 - X_i)ig] \ &= \sum_{ ext{e} \in \pi_i} \widehat{x}_i + \sum_{ ext{e} \in \sigma_i} (1 - \widehat{x}_i) \leq \widehat{w}. \end{aligned}$$

 $\widehat{\boldsymbol{w}}$: Fractional congestion (from LP solution).

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Minimizing congestion - continued

- ② Y_e is just a sum of independent 0/1 random variables!
- Ohernoff inequality tells us sum can not be too far from expectation!

Minimizing congestion - continued

- $ext{ 2 } Y_{ ext{e}} ext{ is just a sum of independent } 0/1 ext{ random variables!}$
- Ohernoff inequality tells us sum can not be too far from expectation!

- $Y_{e} = \sum_{e \in \pi_{i}} X_{i} + \sum_{e \in \sigma_{i}} (1 X_{i}).$
- Chernoff inequality tells us sum can not be too far from expectation!

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By Chernoff inequality:

$$\Pr\big[Y_{\mathsf{e}} \geq (1+\delta)\alpha_{\mathsf{e}}\big] \leq \exp\!\left(-\frac{\alpha_{\mathsf{e}}\delta^2}{4}\right) \leq \exp\!\left(-\frac{\widehat{w}\delta^2}{4}\right).$$

② Let $\delta = \sqrt{\frac{400}{\widehat{w}}} \ln t$. We have that

$$ext{Pr} \Big[Y_{ ext{e}} \geq (1+\delta) lpha_{ ext{e}} \Big] \leq ext{exp} \Big(-rac{\delta^2 \widehat{w}}{4} \Big) \leq rac{1}{t^{100}},$$

- t: Number of pairs, n: Number of vertices in G.

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- $\textbf{ 0} \ \ \text{If} \ t \geq n^{1/50} \implies \forall \ \text{edges in graph congestion} \leq (1+\delta)\widehat{w}.$
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② Play with the numbers. If t=n, and $\widehat{w} \geq \sqrt{n}$. Then, the solution has congestion larger than the optimal solution by a factor of

$$1+\delta=1+\sqrt{\frac{20}{\widehat{w}}\ln t}\leq 1+\frac{\sqrt{20\ln n}}{n^{1/4}},$$

which is of course extremely close to 1, if n is sufficiently large

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Theorem

- **1 G**: Graph **n** vertices.
- $(s_1,t_1),\ldots,(s_t,t_t)$: pairs o vertices
- $oldsymbol{0}$ $oldsymbol{\pi}_i, oldsymbol{\sigma}_i$: two different paths connecting s_i to t_i
- \bigcirc \widehat{w} : Fractional congestion at least $n^{1/2}$.
- opt: Congestion of optimal solution
- $exttt{ o} \implies$ In polynomial time (LP solving time) choose paths
 - **1** congestion \forall edges: $\leq (1 + \delta)$ opt
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- Assume $\widehat{\boldsymbol{w}}$ is a constant.
- 2 Can get a better bound by using the Chernoff inequality in its more general form.
- ③ set $\delta = c \ln t / \ln \ln t$, where c is a constant. For $\mu = \alpha_{\rm e}$, we have that

$$egin{split} \Prig[Y_{\mathsf{e}} &\geq (1+\delta)\muig] &\leq \left(rac{e^{\delta}}{(1+\delta)^{1+\delta}}
ight)^{\mu} \ &= \expigg(\muig(\delta-(1+\delta)\ln(1+\delta)ig)igg) \ &= \expigg(-\mu c'\ln tigg) \leq rac{1}{t^{O(1)}}, \end{split}$$

where c^\prime is a constant that depends on c and grows if c grows.

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- 3 algorithm outputs a solution with congestion $O(\log t/\log\log t)$, and this holds with high probability

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21.4: Reminder about Chernoff inequality

21.4.1: The Chernoff Bound — General Case

Chernoff inequality

Problem

Let $X_1, \ldots X_n$ be n independent Bernoulli trials, where

$$ext{Pr}ig[X_i=1ig]=p_i, \qquad ext{Pr}ig[X_i=0ig]=1-p_i, \ Y=\sum_i X_i, \qquad ext{and} \qquad \mu= ext{E}ig[Yig].$$

We are interested in bounding the probability that $Y \geq (1+\delta)\mu$.

Chernoff inequality

Theorem (Chernoff inequality)

For any $\delta > 0$,

$$ext{Pr}ig[Y>(1+\delta)\muig]$$

Or in a more simplified form, for any $\delta \leq 2e-1$,

$$\Pr[Y > (1+\delta)\mu] < \exp(-\mu\delta^2/4),$$

and

$$\Prig[Y>(1+\delta)\muig]<2^{-\mu(1+\delta)},$$

for $\delta \geq 2e-1$.

More Chernoff...

Theorem

Under the same assumptions as the theorem above, we have

$$ext{Pr} \Big[Y < (1-\delta) \mu \Big] \leq \exp igg(-\mu rac{\delta^2}{2} igg).$$

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