HW 9 (due Monday, 6pm, November 16, 2015)

NEW CS 473: Theory II, Fall 2015 Version: **1.31**

Collaboration Policy: This homework can be worked in groups of up to three students.
Submission is online on moodle.

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1. (50 PTS.) Many cover problem.

Let $(X, \mathcal{F})$ be a set system with $n = |X|$ elements, and $m = |\mathcal{F}|$ sets. Furthermore, for every elements $u \in X$, there is a positive integer c_u. In the ManyCover problem, you need to find a minimum number of sets $\mathcal{G} \subseteq \mathcal{F}$, such that every element of $u \in X$ is covered at least c_u times. (You are not allowed to use the same set more than once in the cover $\mathcal{G}$.)

(A) (10 PTS.) Let $y_1, \ldots, y_n$ be numbers in $[0, 1]$, such that $t = \sum_{i=1}^{n} y_i \geq 3$. Let Y_i be a random variable that is one with probability y_i (and zero otherwise), for all i. Prove, that $\mathbf{Pr}[t/2 \leq \sum_i Y_i \leq 3t/2] \geq 1 - f(t)$, where $f(t)$ is a function that goes to zero as t increases (the smaller the $f(t)$ is, the better your solution is).

(B) (20 PTS.) Describe in detail a polynomial approximation algorithms that provides a $O(\log n)$ approximation to the optimal solution for this problem (as usual, you can assume that solving a polynomially sized LP takes polynomial time). (Hint: See the algorithm provided in class for set Cover.)

(C) (20 PTS.) Provide a polynomial time algorithm, that provides a $O(1)$ approximation to the problem, if we know that $c_u \geq \log n$, for all $u \in X$.

2. (50 PTS.) Independent set via interference.

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph with n vertices, and m edges. Assume we have a feasible solution to the natural independent set for G:

$$
\begin{aligned}
\max \quad & \sum_{v \in \mathsf{V}} x_v \\
\text{s.t.} \quad & x_v + x_u \leq 1 && \forall uv \in \mathsf{E} \\
& x_u \geq 0 && \forall u \in \mathsf{V}.
\end{aligned}
$$

This solution assigns the value $\widehat{x_v}$ to x_v, for all v. Furthermore, assume that $\alpha = \sum_{v \in \mathsf{V}} \widehat{x_v}$ and, importantly, $\sum_{uv \in \mathsf{E}} \widehat{x_u}\widehat{x_v} \leq \alpha/8$.

(A) (10 PTS.) Let S be a subset of the vertices of the graph being generated by picking (independently) each vertex $u \in \mathsf{V}$ to be in S with probability $\widehat{x_u}$.
Prove, that with probability at least $9/10$, we have $|S| \geq \alpha/2$ (you can safely assume that $\alpha \geq n_0$, where n_0 is a sufficiently large constant).

(B) (20 PTS.) Let G_S be the induced subgraph of G on S. Prove that $\mathbf{Pr}\Big[|\mathsf{E}(\mathsf{G}_S)| \geq \alpha/4\Big] \leq 1/2$.

(C) (20 PTS.) Present an algorithm, as fast as possible, that outputs an independent set in G of size at least $c\alpha$, where $c > 0$ is some fixed constant. What is the running time of your algorithm? What is the value of c for your algorithm?