Polynomial Time Reductions

Lecture 20

November 11, 2014

Part I

Introduction to Reductions

Subset sum and Partition?

Problem: Subset Sum

> **Instance:** S - set of positive integers, t: - an integer number (target).
> **Question:** Is there a subset $X \subseteq S$ such that $\sum_{x \in X} x = t$?

Problem: Partition

> **Instance:** A set S of n numbers.
> **Question:** Is there a subset $T \subseteq S$ s.t. $\sum_{t \in T} t = \sum_{s \in S \setminus T} s$?

Assume that we can solve **Subset Sum** in polynomial time, then we can solve **Partition** in polynomial time. This statement is

(A) True.

(B) Mostly true.

(C) False.

(D) Mostly false.

II: Partition and subset sum?

Problem: Partition

> **Instance:** A set $\mathbf{S}$ of $\mathbf{n}$ numbers.
> **Question:** Is there a subset $\mathbf{T} \subseteq \mathbf{S}$ s.t. $\sum_{\mathbf{t} \in \mathbf{T}} \mathbf{t} = \sum_{\mathbf{s} \in \mathbf{S} \setminus \mathbf{T}} \mathbf{s}$?

Problem: Subset Sum

> **Instance:** $\mathbf{S}$ - set of positive integers, $\mathbf{t}$: - an integer number (target).
> **Question:** Is there a subset $\mathbf{X} \subseteq \mathbf{S}$ such that $\sum_{\mathbf{x} \in \mathbf{X}} \mathbf{x} = \mathbf{t}$?

Assume that we can solve **Partition** in polynomial time, then we can solve **Subset Sum** in polynomial time. This statement is

- **(A)** True.
- **(B)** Mostly true.
- **(C)** False.
- **(D)** Mostly false.

III: Partition and Halting?

Problem: Halting

Instance: $\mathbf{P}$: Program, $\mathbf{I}$: Input.
Question: Does $\mathbf{P}$ stop on the input $\mathbf{I}$?

Problem: Partition

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Assume that we can solve **Halting** in polynomial time, then we can solve **Partition** in polynomial time. This statement is

(A) True.

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IV: Halting and Partition?

Problem: Partition

> **Instance:** A set $\mathbf{S}$ of $\mathbf{n}$ numbers.
> **Question:** Is there a subset $\mathbf{T} \subseteq \mathbf{S}$ s.t. $\sum_{t \in T} t = \sum_{s \in S \setminus T} s$?

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Assume that we can solve **Partition** in polynomial time, then we can solve **Halting** in polynomial time. This statement is

- **(A)** True.
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What we know...

1. **Partition $\approx_P$ Subset sum**.
2. **Halting** is way way way way way way **harder**.

Reductions

A reduction from Problem **X** to Problem **Y** means (informally) that if we have an algorithm for Problem **Y**, we can use it to find an algorithm for Problem **X**.

Using Reductions

1. We use reductions to find algorithms to solve problems.

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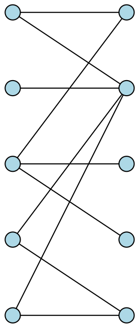
Also, the right reductions might win you a million dollars!

Example 1: Bipartite Matching and Flows

How do we solve the Bipartite Matching Problem?

Given a bipartite graph $\mathbf{G} = (\mathbf{U} \cup \mathbf{V}, \mathbf{E})$ and number $\mathbf{k}$, does $\mathbf{G}$ have a matching of size $\geq \mathbf{k}$?



Solution

Reduce it to **Max-Flow**. G has a matching of size $\geq \mathbf{k}$ iff there is a flow from $\mathbf{s}$ to $\mathbf{t}$ of value $\geq \mathbf{k}$.

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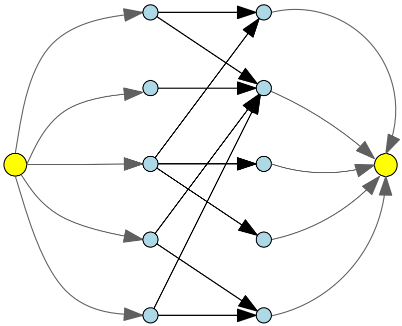

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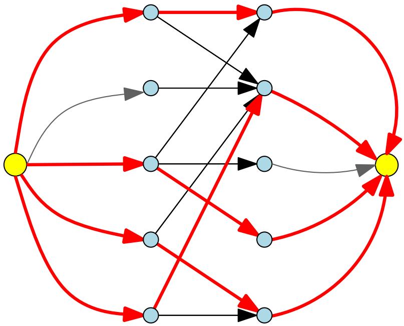

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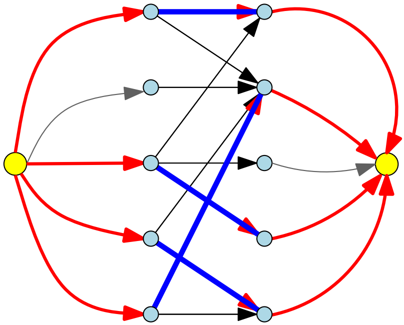

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Types of Problems

Decision, Search, and Optimization

1. **Decision problem**. Example: given $\mathbf{n}$, is $\mathbf{n}$ prime?.
2. **Search problem**. Example: given $\mathbf{n}$, find a factor of $\mathbf{n}$ if it exists.
3. **Optimization problem**. Example: find the smallest prime factor of $\mathbf{n}$.

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Optimization and Decision problems

For max flow...

Problem (Max-Flow optimization version)

Given an instance G of network flow, find the maximum flow between **s** *and* **t**.

Problem (Max-Flow decision version)

Given an instance G of network flow and a parameter **K**, *is there a flow in G, from* **s** *to* **t**, *of value at least* **K**?

While using reductions and comparing problems, we typically work with the decision versions. Decision problems have Yes/No answers. This makes them easy to work with.

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Problems vs Instances

1. A problem Π consists of an **infinite** collection of inputs $\{I_1, I_2, \ldots, \}$. Each input is referred to as an instance.
2. The size of an instance I is the number of bits in its representation.
3. For an instance I, $sol(I)$ is a set of feasible solutions to I.
4. For optimization problems each solution $s \in sol(I)$ has an associated value.

Examples

Example

An instance of **Bipartite Matching** is a bipartite graph, and an integer **k**. The solution to this instance is "YES" if the graph has a matching of size $\geq$ **k**, and "NO" otherwise.

Example

An instance of **Max-Flow** is a graph **G** with edge-capacities, two vertices **s, t**, and an integer **k**. The solution to this instance is "YES" if there is a flow from **s** to **t** of value $\geq$ **k**, else 'NO".

What is an algorithm for a decision Problem **X**?

It takes as input an instance of **X**, and outputs either "YES" or "NO".

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Encoding an instance into a string

1. **I**; Instance of some problem.
2. **I** can be fully and precisely described (say in a text file).
3. Resulting text file is a binary string.
4. $\implies$ Any input can be interpreted as a binary string **S**.
5. ... Running time of algorithm: Function of length of **S** (i.e., **n**).

Decision Problems and Languages

1. A finite alphabet Σ. Σ^* is set of all finite strings on Σ.
2. A language L is simply a subset of Σ^*; a set of strings.

For every language L there is an associated decision problem Π_L and conversely, for every decision problem Π there is an associated language L_Π.

1. Given L, Π_L is the following decision problem: Given $x \in \Sigma^*$, is $x \in L$? Each string in Σ^* is an instance of Π_L and L is the set of instances for which the answer is YES.
2. Given Π the associated language is

$$L_\Pi = \left\{ I \;\middle|\; I \text{ is an instance of } \Pi \text{ for which answer is YES} \right\}.$$

Thus, decision problems and languages are used interchangeably.

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Example

1. The decision problem **Primality**, and the language

$$\mathbf{L} = \left\{ \#\mathbf{p} \;\middle|\; \mathbf{p} \text{ is a prime number} \right\}.$$

Here $\#\mathbf{p}$ is the string in base $\mathbf{10}$ representing $\mathbf{p}$.

2. **Bipartite** (is given graph is bipartite. The language is

$$\mathbf{L} = \left\{ \boldsymbol{\mathcal{S}}(\mathsf{G}) \;\middle|\; \mathsf{G} \text{ is a bipartite graph} \right\}.$$

Here $\boldsymbol{\mathcal{S}}(\mathsf{G})$ is the string encoding the graph G.

Are regular languages good?

Let $\mathbf{L}$ be a regular language. Then the decision problem associated with $\mathbf{L}$ can be solved in

- **(A)** Constant time.
- **(B)** Linear time.
- **(C)** Quadratic time.
- **(D)** Exponential time.
- **(E)** Doubly exponential time (i.e., 2^{2^n}).
- **(F)** Octly exponential time (i.e., $2^{2^{2^{2^{2^{2^{2^{2^n}}}}}}}$).

Reductions, revised.

For decision problems **X, Y**, a **reduction from X to Y** is:

1. An algorithm ...
2. Input: $\mathbf{I_X}$, an instance of **X**.
3. Output: $\mathbf{I_Y}$ an instance of **Y**.
4. Such that:

$$\boxed{\mathbf{I_Y} \text{ is YES instance of } \mathbf{Y}} \iff \boxed{\mathbf{I_X} \text{ is YES instance of } \mathbf{X}}$$

There are other kinds of reductions.

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Using reductions to solve problems

1. $\mathcal{R}$: Reduction $\mathbf{X} \rightarrow \mathbf{Y}$
2. $\mathcal{A}_{\mathbf{Y}}$: algorithm for $\mathbf{Y}$:
3. $\implies$ New algorithm for $\mathbf{X}$:

```
A_X(I_X):
        // I_X: instance of X.
        I_Y ⇐ R(I_X)
        return A_Y(I_Y)
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If $\mathcal{R}$ and $\mathcal{A}_{\mathbf{Y}}$ polynomial-time $\implies$ $\mathcal{A}_{\mathbf{X}}$ polynomial-time.

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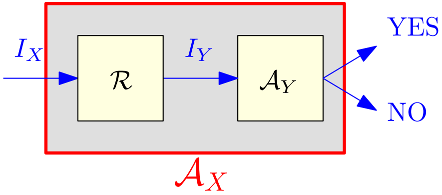

If $\mathcal{R}$ and $\mathcal{A}_\mathbf{Y}$ polynomial-time $\implies$ $\mathcal{A}_\mathbf{X}$ polynomial-time.

Comparing Problems

1. "Problem **X** is no harder to solve than Problem **Y**".
2. If Problem **X** reduces to Problem **Y** (we write **X** $\leq$ **Y**), then **X** cannot be harder to solve than **Y**.
3. **Bipartite Matching** $\leq$ **Max-Flow**.
   Bipartite Matching cannot be harder than **Max-Flow**.
4. Equivalently,
   Max-Flow is at least as hard as **Bipartite Matching**.
5. **X** $\leq$ **Y**:
   1. **X** is no harder than **Y**, or
   2. **Y** is at least as hard as **X**.

Part II

Examples of Reductions

Independent Sets and Cliques

Given a graph $\mathbf{G}$, a set of vertices $\mathbf{V'}$ is:

1. **independent set**: no two vertices of $\mathbf{V'}$ connected by an edge.

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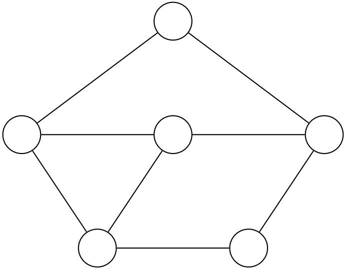

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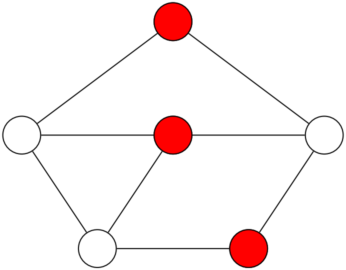

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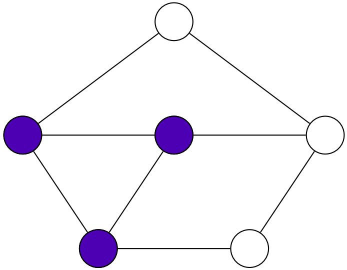

The **Independent Set** and **Clique** Problems

Problem: Independent Set

Instance: A graph G and an integer **k**.
Question: Does G has an independent set of size $\geq$ **k**?

Problem: Clique

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3. and returns $\mathbf{I_Y}$, an instance of $\mathbf{Y}$ as output ...
4. such that the solution (YES/NO) to $\mathbf{I_Y}$ is the same as the solution to $\mathbf{I_X}$.

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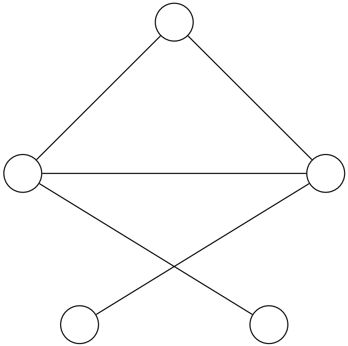

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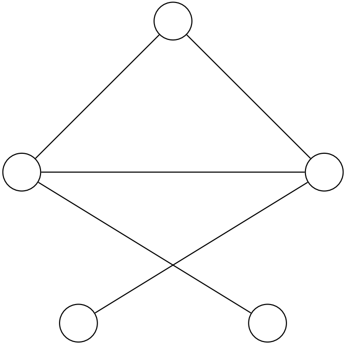
Convert $\mathbf{G}$ to $\overline{\mathbf{G}}$, in which $(\mathbf{u}, \mathbf{v})$ is an edge iff $(\mathbf{u}, \mathbf{v})$ is not an edge of $\mathbf{G}$. ($\overline{\mathbf{G}}$ is the *complement* of $\mathbf{G}$.)
We use $\overline{\mathbf{G}}$ and $\mathbf{k}$ as the instance of **Clique**.



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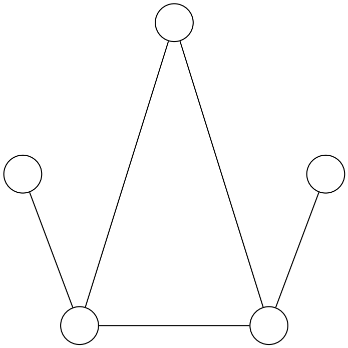

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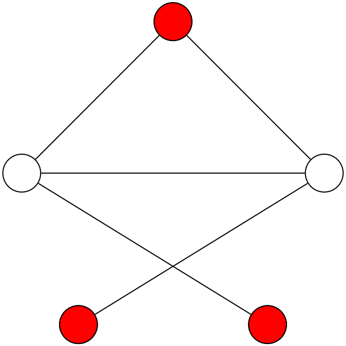

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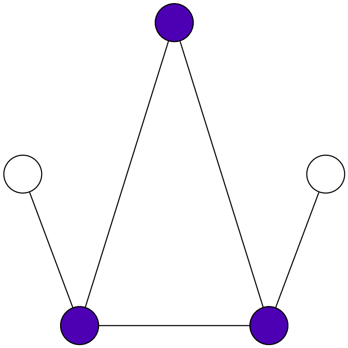

Independent Set and **Clique**

1. **Independent Set** $\leq$ **Clique**.
   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
3. **Clique** is *at least as hard as* **Independent Set**.
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1. **Independent Set** $\leq$ **Clique**.
   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
3. **Clique** is *at least as hard as* **Independent Set**.
4. Also... **Independent Set** is *at least as hard as* **Clique**.

Independent and Clique

Assume you can solve the **Clique** problem in $\mathbf{T(n)}$ time. Then you can solve the **Independent Set** problem in

(A) $\mathbf{O(T(n))}$ time.

(B) $\mathbf{O(n \log n + T(n))}$ time.

(C) $\mathbf{O(n^2 T(n^2))}$ time.

(D) $\mathbf{O(n^4 T(n^4))}$ time.

(E) $\mathbf{O(n^2 + T(n^2))}$ time.

(F) Does not matter - all these are polynomial if $\mathbf{T(n)}$ is polynomial, which is good enough for our purposes.

DFAs and NFAs

DFAs (Remember 373?) are automata that accept regular languages. NFAs are the same, except that they are non-deterministic, while DFAs are deterministic.

Every NFA can be converted to a DFA that accepts the same language using the subset construction.

(How long does this take?)
The smallest DFA equivalent to an NFA with n states may have $\approx 2^n$ states.

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DFA Universality

A DFA $\mathbf{M}$ is universal if it accepts every string.
That is, $\mathbf{L(M) = \Sigma^*}$, the set of all strings.

Problem (DFA universality)

Input: *A DFA* $\mathbf{M}$.
Goal: *Is* $\mathbf{M}$ *universal?*

How do we solve **DFA Universality**?
We check if $\mathbf{M}$ has *any* reachable non-final state.
Alternatively, minimize $\mathbf{M}$ to obtain $\mathbf{M'}$ and see if $\mathbf{M'}$ has a single state which is an accepting state.

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Reduce it to **DFA Universality**?
Given an NFA **N**, convert it to an equivalent DFA **M**, and use the **DFA Universality** Algorithm.

The reduction takes exponential time!

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We say that an algorithm is **efficient** if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

If we have a polynomial-time reduction from problem **X** to problem **Y** (we write $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$), and a poly-time algorithm $\mathcal{A}_{\mathbf{Y}}$ for **Y**, we have a polynomial-time/efficient algorithm for **X**.

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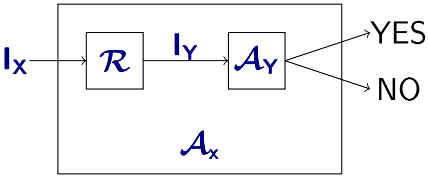

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A polynomial time reduction from a *decision* problem $\mathbf{X}$ to a *decision* problem $\mathbf{Y}$ is an *algorithm* $\mathcal{A}$ that has the following properties:

1. given an instance $\mathbf{I_X}$ of $\mathbf{X}$, $\mathcal{A}$ produces an instance $\mathbf{I_Y}$ of $\mathbf{Y}$
2. $\mathcal{A}$ runs in time polynomial in $|\mathbf{I_X}|$.
3. Answer to $\mathbf{I_X}$ YES *iff* answer to $\mathbf{I_Y}$ is YES.

Proposition

If $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$ *then a polynomial time algorithm for* $\mathbf{Y}$ *implies a polynomial time algorithm for* $\mathbf{X}$.

Such a reduction is called a **Karp reduction**. Most reductions we will need are Karp reductions.

Reductions again...

Let **X** and **Y** be two decision problems, such that **X** can be solved in polynomial time, and $\mathbf{X} \leq_{\mathbf{P}} \mathbf{Y}$. Then

- **(A)** **Y** can be solved in polynomial time.
- **(B)** **Y** can NOT be solved in polynomial time.
- **(C)** If **Y** is hard then **X** is also hard.
- **(D)** None of the above.
- **(E)** All of the above.

Polynomial-time reductions and hardness

For decision problems **X** and **Y**, if **X** $\leq_P$ **Y**, and **Y** has an efficient algorithm, **X** has an efficient algorithm.

If you believe that **Independent Set** does not have an efficient algorithm, why should you believe the same of **Clique**?

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Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from $\mathbf{X}$ to $\mathbf{Y}$. Then for any instance $\mathbf{I_X}$ of $\mathbf{X}$, the size of the instance $\mathbf{I_Y}$ of $\mathbf{Y}$ produced from $\mathbf{I_X}$ by $\mathcal{R}$ is polynomial in the size of $\mathbf{I_X}$.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input $\mathbf{I_X}$ of size $|\mathbf{I_X}|$ it runs in time $\mathbf{p}(|\mathbf{I_X}|)$ for some polynomial $\mathbf{p}()$.
$\mathbf{I_Y}$ is the output of $\mathcal{R}$ on input $\mathbf{I_X}$.
$\mathcal{R}$ can write at most $\mathbf{p}(|\mathbf{I_X}|)$ bits and hence $|\mathbf{I_Y}| \leq \mathbf{p}(|\mathbf{I_X}|)$. □

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

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Transitivity of Reductions

Proposition

$\mathbf{X} \leq_P \mathbf{Y}$ *and* $\mathbf{Y} \leq_P \mathbf{Z}$ *implies that* $\mathbf{X} \leq_P \mathbf{Z}$.

Note: $\mathbf{X} \leq_P \mathbf{Y}$ does not imply that $\mathbf{Y} \leq_P \mathbf{X}$ and hence it is very important to know the FROM and TO in a reduction.

To prove $\mathbf{X} \leq_P \mathbf{Y}$ you need to show a reduction FROM $\mathbf{X}$ TO $\mathbf{Y}$
In other words show that an algorithm for $\mathbf{Y}$ implies an algorithm for $\mathbf{X}$.

Vertex Cover

Given a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, a set of vertices $\mathbf{S}$ is:

1. A **vertex cover** if every $\mathbf{e} \in \mathbf{E}$ has at least one endpoint in $\mathbf{S}$.

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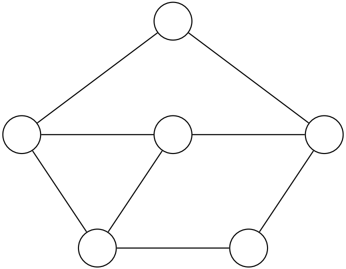

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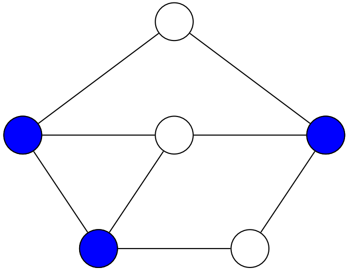

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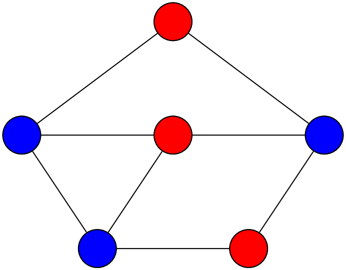

The **Vertex Cover** Problem

Problem (**Vertex Cover**)

Input: *A graph G and integer* $\mathbf{k}$.
Goal: *Is there a vertex cover of size* $\leq \mathbf{k}$ *in* G*?*

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Relationship between...

Vertex Cover and Independent Set

Proposition

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a graph. $\mathbf{S}$ is an independent set if and only if $\mathbf{V} \setminus \mathbf{S}$ is a vertex cover.

Proof.

($\Rightarrow$) Let $\mathbf{S}$ be an independent set

1. Consider any edge $\mathbf{uv} \in \mathbf{E}$.
2. Since $\mathbf{S}$ is an independent set, either $\mathbf{u} \notin \mathbf{S}$ or $\mathbf{v} \notin \mathbf{S}$.
3. Thus, either $\mathbf{u} \in \mathbf{V} \setminus \mathbf{S}$ or $\mathbf{v} \in \mathbf{V} \setminus \mathbf{S}$.
4. $\mathbf{V} \setminus \mathbf{S}$ is a vertex cover.

($\Leftarrow$) Let $\mathbf{V} \setminus \mathbf{S}$ be some vertex cover:

1. Consider $\mathbf{u}, \mathbf{v} \in \mathbf{S}$
2. $\mathbf{uv}$ is not an edge of G, as otherwise $\mathbf{V} \setminus \mathbf{S}$ does not cover $\mathbf{uv}$.
3. $\implies$ $\mathbf{S}$ is thus an independent set. □

Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$
3. (G, k) is an instance of **Independent Set** , and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
4. Therefore, **Independent Set** $\leq_P$ **Vertex Cover**. Also **Vertex Cover** $\leq_P$ **Independent Set**.

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What about edge cover?

Problem: Edge Cover

Instance: A graph G and integer **k**. **Question:** Is there a subset of **k** edges such that all the vertices of G are adjacent to one of these edges.

We have that:

(A) **Edge Cover** is polynomially equivalent to **Independent Set**.

(B) **Edge Cover** is polynomially equivalent to **Vertex Cover**.

(C) **Edge Cover** is polynomially equivalent to **Clique**.

(D) **Edge Cover** is polynomially equivalent to **3 COLORING**.

(E) None of the above.

Can you reduce between these problems

Problem: 2SAT

Instance: **F**: a 2CNF formula.
Question: Is there a satisfying assignment to **F**?

Problem: Max Flow

Instance: **G, s, t, k**: Instance of network flow.
Question: Is there a valid flow in **G** from **s** to **t** of value larger than **k**?

(A) **2SAT** $\leq_P$ **Max Flow**.

(B) **Max Flow** $\leq_P$ **2SAT**.

(C) **2SAT** $\leq_P$ **Max Flow** and **Max Flow** $\leq_P$ **2SAT**.

(D) There is NO polynomial time reduction from **2SAT** to **Max Flow**, or vice versa.

(E) All your reduction belong to us.

A problem of Languages

Suppose you work for the United Nations. Let **U** be the set of all languages spoken by people across the world. The United Nations also has a set of translators, all of whom speak English, and some other languages from **U**.

Due to budget cuts, you can only afford to keep **k** translators on your payroll. Can you do this, while still ensuring that there is someone who speaks every language in **U**?

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The Set Cover Problem

Problem (Set Cover)

Input: *Given a set* $\mathbf{U}$ *of* $\mathbf{n}$ *elements, a collection* $\mathbf{S_1, S_2, \ldots S_m}$ *of subsets of* $\mathbf{U}$*, and an integer* $\mathbf{k}$.

Goal: *Is there a collection of at most* $\mathbf{k}$ *of these sets* $\mathbf{S_i}$ *whose union is equal to* $\mathbf{U}$*?*

Example

Let $\mathbf{U} = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathbf{k} = 2$ with

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1. Number k for the **Set Cover** instance is the same as the number k given for the **Vertex Cover** instance.
2. $U = E$.
3. We will have one set corresponding to each vertex; $S_v = \{e \mid e \text{ is incident on } v\}$.

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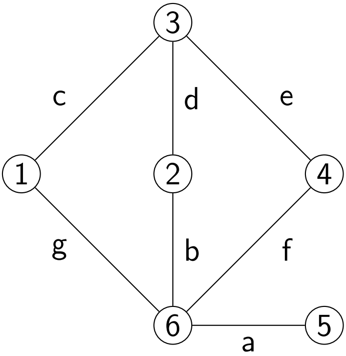
Observe that G has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

Vertex Cover $\leq_P$ Set Cover: Example



Let $\mathbf{U} = \{a, b, c, d, e, f, g\}$, $\mathbf{k} = \mathbf{2}$ with

$\mathbf{S_1} = \{c, g\}$ $\quad$ $\mathbf{S_2} = \{b, d\}$
$\mathbf{S_3} = \{c, d, e\}$ $\quad$ $\mathbf{S_4} = \{e, f\}$
$\mathbf{S_5} = \{a\}$ $\quad$ $\mathbf{S_6} = \{a, b, f, g\}$

$\{S_3, S_6\}$ is a set cover

$\{3, 6\}$ is a vertex cover

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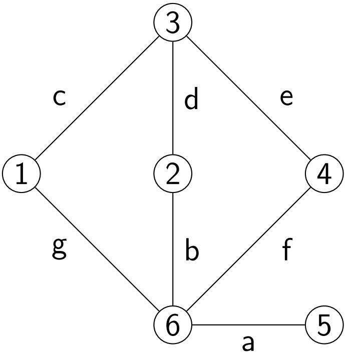

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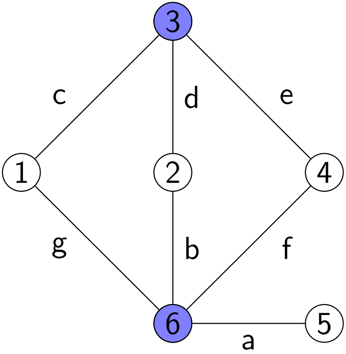

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Proving Reductions

To prove that $\mathbf{X} \leq_P \mathbf{Y}$ you need to give an algorithm $\mathcal{A}$ that:

1. Transforms an instance $\mathbf{I_X}$ of $\mathbf{X}$ into an instance $\mathbf{I_Y}$ of $\mathbf{Y}$.
2. Satisfies the property that answer to $\mathbf{I_X}$ is YES iff $\mathbf{I_Y}$ is YES.
    1. typical easy direction to prove: answer to $\mathbf{I_Y}$ is YES if answer to $\mathbf{I_X}$ is YES
    2. typical difficult direction to prove: answer to $\mathbf{I_X}$ is YES if answer to $\mathbf{I_Y}$ is YES (equivalently answer to $\mathbf{I_X}$ is NO if answer to $\mathbf{I_Y}$ is NO).
3. Runs in **polynomial** time.

Vertex cover and Set cover?

Consider the statement: **Set Cover $\leq_P$ Vertex Cover**.
This statement is

(A) correct.

(B) correct (although the reduction seen is in the other direction - so not clear why this is correct).

(C) incorrect.

(D) incorrect (the reduction seen is in the other direction!)

Example of incorrect reduction proof

Try proving **Matching** $\leq_P$ **Bipartite Matching** via following reduction:

1. Given graph $G = (V, E)$ obtain a bipartite graph $G' = (V', E')$ as follows.
   1. Let $V_1 = \{u_1 \mid u \in V\}$ and $V_2 = \{u_2 \mid u \in V\}$. We set $V' = V_1 \cup V_2$ (that is, we make two copies of V)
   2. $E' = \left\{ u_1 v_2 \,\middle|\, u \neq v \text{ and } uv \in E \right\}$
2. Given G and integer k the reduction outputs G' and k.

Example

"Proof"

Claim

Reduction is a poly-time algorithm. If $\mathbf{G}$ *has a matching of size* $\mathbf{k}$ *then* $\mathbf{G'}$ *has a matching of size* $\mathbf{k}$.

Proof.

Exercise. □

Claim

If $\mathbf{G'}$ *has a matching of size* $\mathbf{k}$ *then* $\mathbf{G}$ *has a matching of size* $\mathbf{k}$.

Incorrect! Why? Vertex $\mathbf{u} \in \mathbf{V}$ has two copies $\mathbf{u_1}$ and $\mathbf{u_2}$ in $\mathbf{G'}$. A matching in $\mathbf{G'}$ may use both copies!

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Summary

We looked at polynomial-time reductions.

Using polynomial-time reductions

1. If $\mathbf{X} \leq_P \mathbf{Y}$, and there is no efficient algorithm for $\mathbf{X}$, there is no efficient algorithm for $\mathbf{Y}$.

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Using polynomial-time reductions

1. If **X** $\leq_P$ **Y**, and we have an efficient algorithm for **Y**, we have an efficient algorithm for **X**.

We looked at some examples of reductions between **Independent Set**, **Clique**, **Vertex Cover**, and **Set Cover**.

Notes

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