Applications of Network Flows

Lecture 18

October 30, 2014

Is the flow always integral?

Let G be an integral instance of network flow (i.e., all numbers are integers). Consider the following statements:

(I) The value of the maximum flow is an integer number.

(II) If $\mathbf{f}$ is a maximum flow, then $\mathbf{f(e)}$ is an integer, for any edge $\mathbf{e} \in E(G)$.

(III) There always exists a max flow $\mathbf{g}$, such that $\mathbf{g}$ is a maximum flow, and $\mathbf{g(e)}$ is an integer, for any edge $\mathbf{e} \in E(G)$.

We have the following:

(A) All the above statements are false.

(B) All the above statements are true.

(C) (I) is true, (II) and (III) are false.

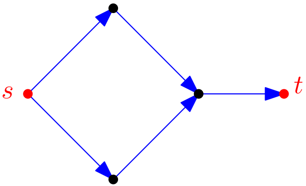
(D) (I) and (II) are true, and (III) is false.

(E) (I) and (III) are true, and (II) is false.

Why max-flow does not have to be integral...

...but the one we compute always is!

Consider the graph with all capacities being one.



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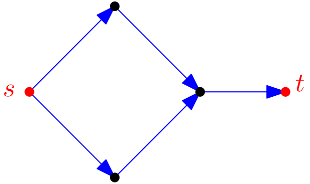
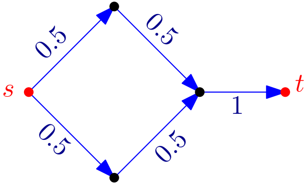


One possible max flow:



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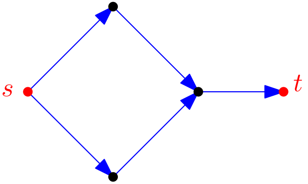

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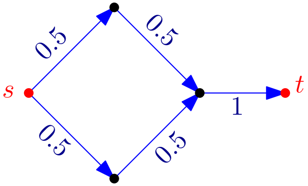
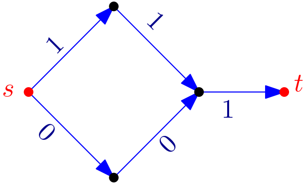


Max flow as computed by **algEdmondsKarp** or **algFordFulkerson**:



Network Flow: Facts to Remember

Flow network: directed graph $\mathbf{G}$, capacities $\mathbf{c}$, source $\mathbf{s}$, sink $\mathbf{t}$.

1. Maximum $\mathbf{s}$-$\mathbf{t}$ flow can be computed:
   1. Using Ford-Fulkerson algorithm in $\mathbf{O(mC)}$ time when capacities are integral and $\mathbf{C}$ is an upper bound on the flow.
   2. Using variant of algorithm, in $\mathbf{O(m^2 \log C)}$ time, when capacities are integral. (Polynomial time.)
   3. Using Edmonds-Karp algorithm, in $\mathbf{O(m^2 n)}$ time, when capacities are rational (strongly polynomial time algorithm).

Network Flow

Even more facts to remember

1. If capacities are integral then there is a maximum flow that is integral and above algorithms give an integral max flow. This is known as **integrality of flow**.
2. Given a flow of value $\mathbf{v}$, can decompose into $\mathbf{O(m + n)}$ flow paths of same total value $\mathbf{v}$. Integral flow implies integral flow on paths.
3. Maximum flow is equal to the minimum cut and minimum cut can be found in $\mathbf{O(m + n)}$ time given any maximum flow.

Paths, Cycles and Acyclicity of Flows

Definition

Given a flow network $G = (V, E)$ and a flow $f : E \to \mathbb{R}^{\geq 0}$ on the edges, the **support** of f is the set of edges $E' \subseteq E$ with non-zero flow on them. That is, $E' = \{e \in E \mid f(e) > 0\}$.

Question: Given a flow f, can there by cycles in its support?

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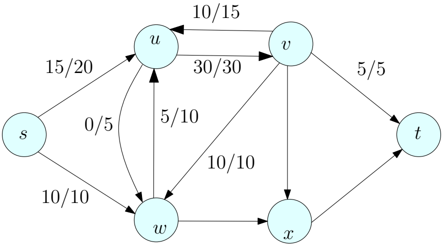

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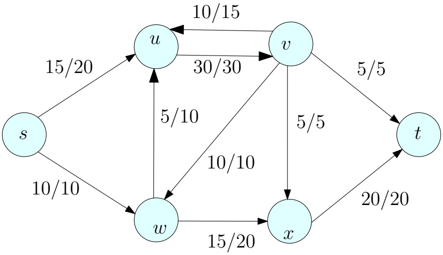

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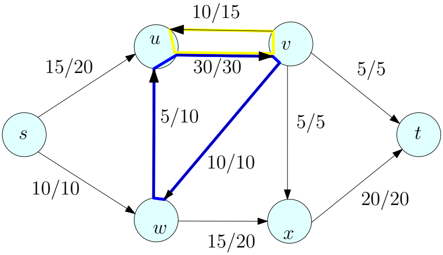


How fast to detect a cycle in the flow

Given a flow network G with $\mathbf{n}$ vertices, and $\mathbf{m}$ edges, and a flow $\mathbf{f}$ on it, then detecting a cycle in the flow can be done in time

(A) $\mathbf{O(m + n)}$.

(B) $\mathbf{O(mC)}$.

(C) $\mathbf{O(mn)}$.

(D) $\mathbf{O(m^2 n)}$.

(E) $\mathbf{O(mn^2)}$.

Acyclicity of Flows

Proposition

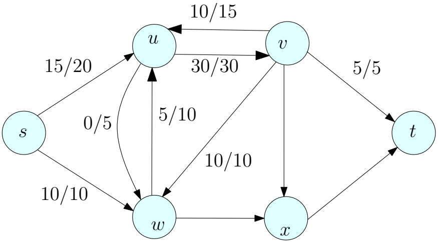
In any flow network, if $\mathbf{f}$ *is a flow then there is another flow* $\mathbf{f'}$ *such that the support of* $\mathbf{f'}$ *is an* acyclic *graph and* $\mathbf{v(f') = v(f)}$*. Further if* $\mathbf{f}$ *is an integral flow then so is* $\mathbf{f'}$*.*

Proof.

1. $\mathbf{E' = \{e \in E \mid f(e) > 0\}}$, support of $\mathbf{f}$.
2. Suppose there is a directed cycle $\mathbf{C}$ in $\mathbf{E'}$
3. Let $\mathbf{e'}$ be the edge in $\mathbf{C}$ with least amount of flow
4. For each $\mathbf{e \in C}$, reduce flow by $\mathbf{f(e')}$. Remains a flow. Why?
5. Flow on $\mathbf{e'}$ is reduced to $\mathbf{0}$.
6. Claim: Flow value from $\mathbf{s}$ to $\mathbf{t}$ does not change. Why?
7. Iterate until no cycles

□

Example



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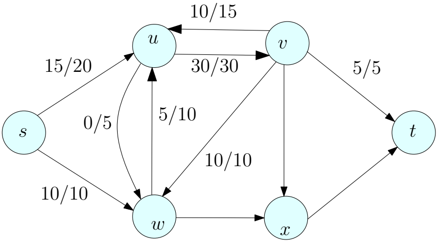
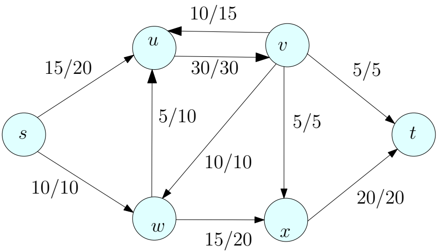
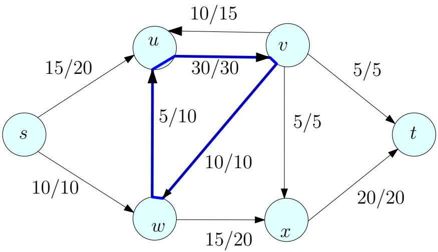


Throw away edge with no flow on it

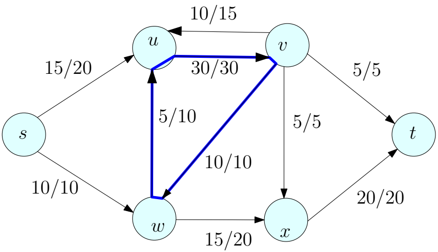
Example





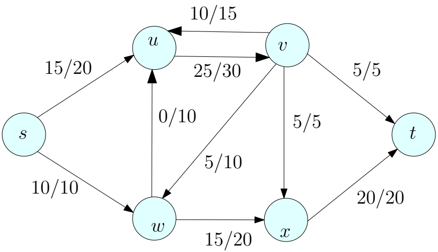
Find a cycle in the support/flow

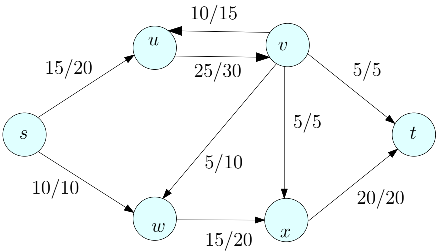
Example



Reduce flow on cycle as much as possible

Example





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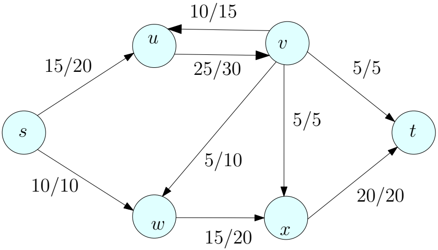

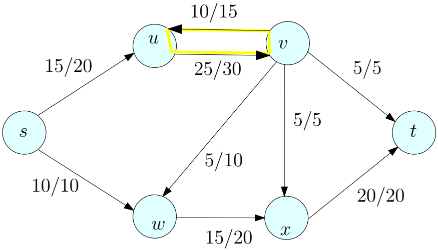

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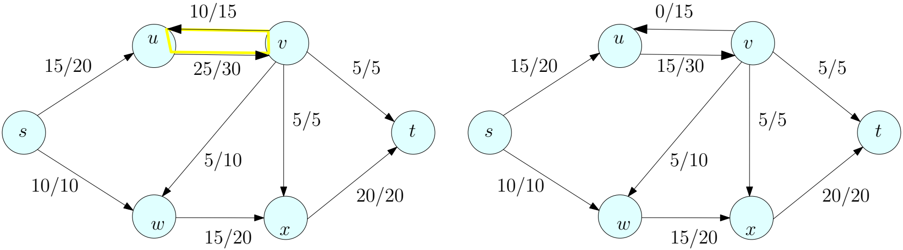

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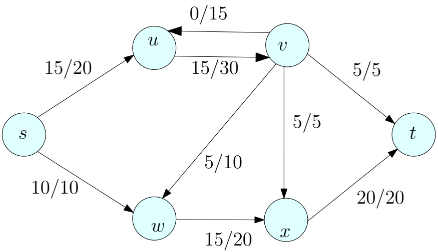

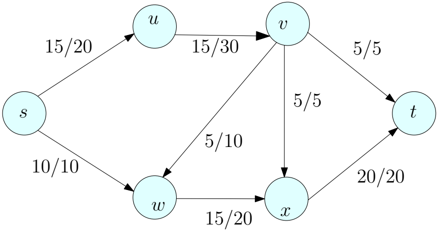

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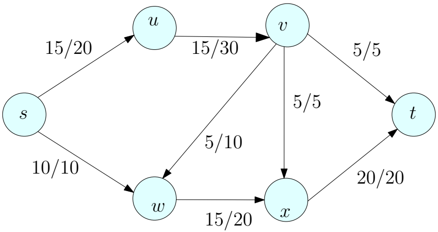
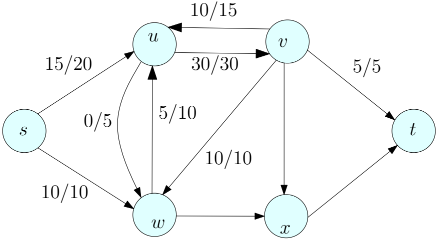


Viola!!! An equivalent flow with no cycles in it. Original flow:



Flow Decomposition

Lemma

Given an edge based flow $\mathbf{f : E \to \mathbb{R}^{\geq 0}}$*, there exists a collection of paths* $\mathcal{P}$ *and cycles* $\mathcal{C}$ *and an assignment of flow to them* $\mathbf{f' : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}^{\geq 0}}$ *such that:*

1. $|\mathcal{P} \cup \mathcal{C}| \leq \mathbf{m}$
2. *for each* $\mathbf{e} \in \mathbf{E}$, $\sum_{\mathbf{P} \in \mathcal{P}: \mathbf{e} \in \mathbf{P}} \mathbf{f'(P)} + \sum_{\mathbf{C} \in \mathcal{C}: \mathbf{e} \in \mathbf{C}} \mathbf{f'(C)} = \mathbf{f(e)}$
3. $\mathbf{v(f)} = \sum_{\mathbf{P} \in \mathcal{P}} \mathbf{f'(P)}$.
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Proof Idea.

1. Remove all cycles as in previous proposition.
2. Next, decompose into paths as in previous lecture.
3. Exercise: verify claims.

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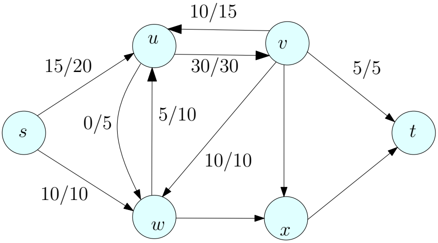
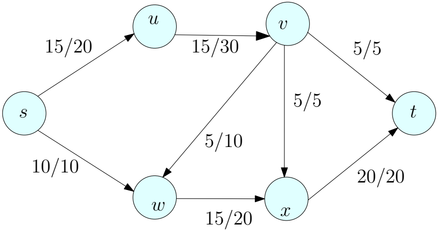
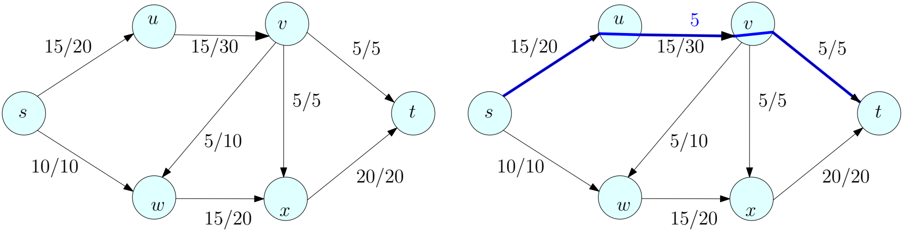




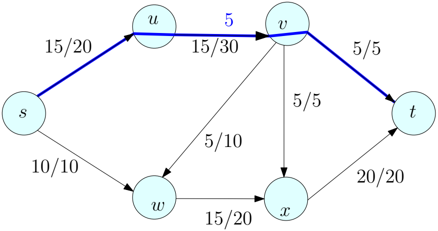
Find cycles as shown before

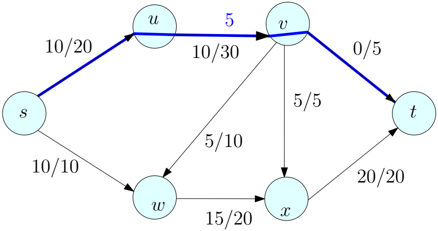
Example



Find a source to sink path, and push max flow along it (5 unites)

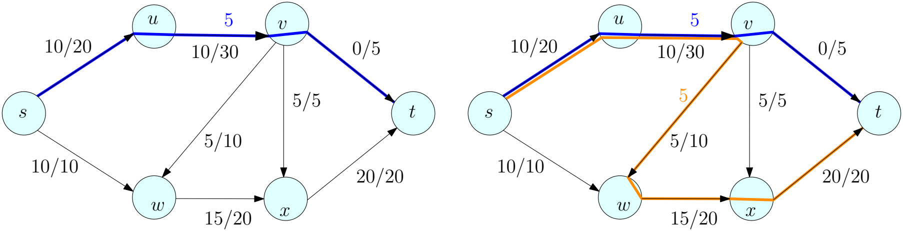
Example





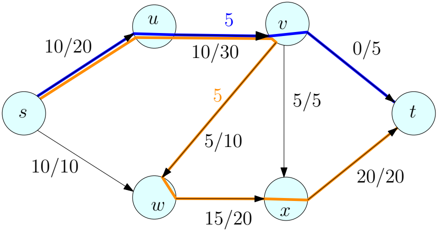
Compute remaining flow

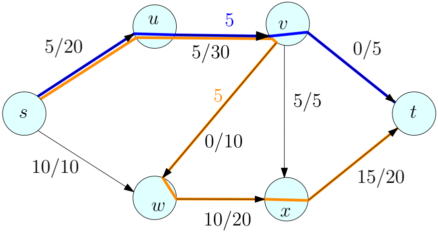
Example



Find a source to sink path, and push max flow along it (5 unites). Edges with **0** flow on them can not be used as they are no longer in the support of the flow.

Example





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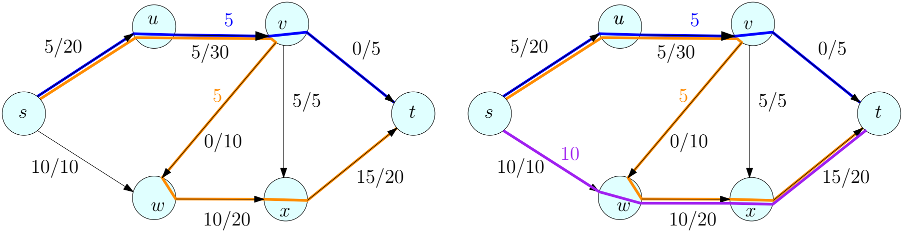

Find a source to sink path, and push max flow along it (10 unites).

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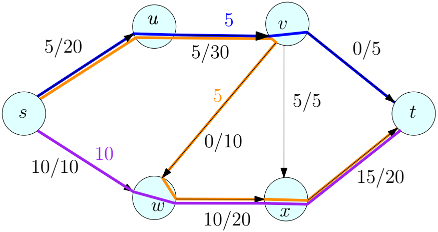

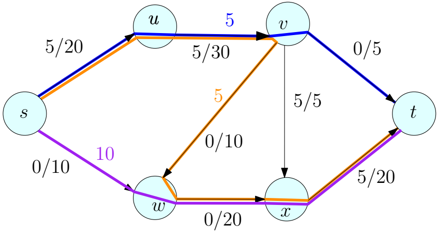

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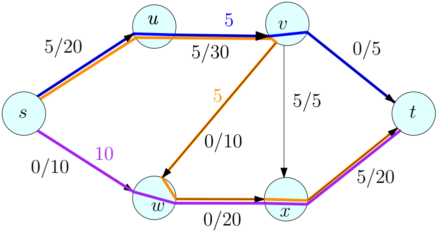

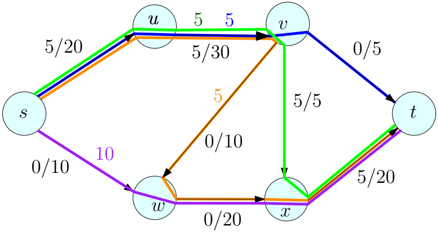

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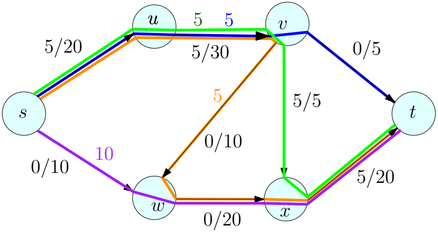

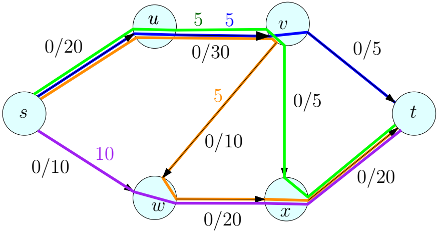

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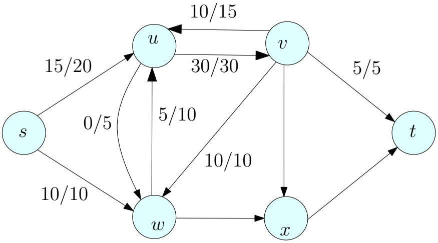

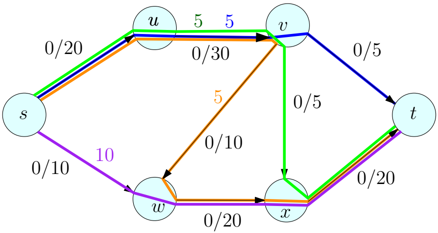

No flow remains in the graph. We fully decomposed the flow into flow on paths. Together with the cycles, we get a decomposition of the original flow into **m** flows on paths and cycles.

Flow Decomposition

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4. *if* $\mathbf{f}$ *is integral then so are* $\mathbf{f}'(\mathbf{P})$ *and* $\mathbf{f}'(\mathbf{C})$ *for all* $\mathbf{P}$ *and* $\mathbf{C}$.

Above flow decomposition can be computed in $\mathbf{O}(\mathbf{m}^2)$ *time.*

Flow decomposition into paths and cycles

Consider an integral flow network G, and two maximum flows $\mathbf{f}$ and $\mathbf{g}$ in G. Assume both $\mathbf{f}$ and $\mathbf{g}$ are acyclic. Let $\mathbf{P_f}$ and $\mathbf{P_g}$ be the decomposition of the two flows into paths. Then:

(A) $\mathbf{P_f = P_g}$ (paths are the same).

(B) $\mathbf{|P_f| = |P_g|}$ (i.e., number of paths is the same).

(C) $\mathbf{|P_f| + |P_g| = m}$.

(D) $\mathbf{|P_f| * |P_g| = nm}$.

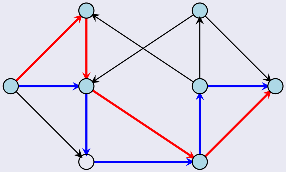
(E) None of the above.

Part I

Network Flow Applications I

Edge-Disjoint Paths in Directed Graphs

Definition



A set of paths is **edge disjoint** if no two paths share an edge.

Problem

Given a directed graph with two special vertices **s** and **t**, find the *maximum* number of edge disjoint paths from **s** to **t**.

Applications: Fault tolerance in routing — edges/nodes in networks can fail. Disjoint paths allow for planning backup routes in case of failures.

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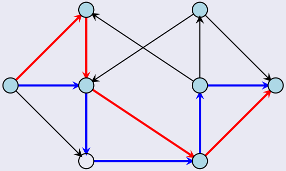

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Reduction to Max-Flow

Problem

Given a directed graph **G** with two special vertices **s** and **t**, find the maximum number of edge disjoint paths from **s** to **t**.

Reduction

Consider **G** as a flow network with edge capacities **1**, and compute max-flow.

Correctness of Reduction

Lemma

If $\mathbf{G}$ *has* $\mathbf{k}$ *edge disjoint paths* $\mathbf{P_1, P_2, \ldots, P_k}$ *then there is an* $\mathbf{s}$*-*$\mathbf{t}$ *flow of value* $\mathbf{k}$ *in* $\mathbf{G}$*.*

Proof.

Set $\mathbf{f(e) = 1}$ if $\mathbf{e}$ belongs to one of the paths $\mathbf{P_1, P_2, \ldots, P_k}$; other-wise set $\mathbf{f(e) = 0}$. This defines a flow of value $\mathbf{k}$. □

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If $\mathbf{G}$ has a flow of value $\mathbf{k}$ then there are $\mathbf{k}$ edge disjoint paths between $\mathbf{s}$ and $\mathbf{t}$.

Proof.

1. Capacities are all $\mathbf{1}$ and hence there is integer flow of value $\mathbf{k}$, that is $\mathbf{f(e) = 0}$ or $\mathbf{f(e) = 1}$ for each $\mathbf{e}$.
2. Decompose flow into paths.
3. Flow on each path is either $\mathbf{1}$ or $\mathbf{0}$.
4. Hence there are $\mathbf{k}$ paths $\mathbf{P_1, P_2, \ldots, P_k}$ with flow of $\mathbf{1}$ each.
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Theorem

The number of edge disjoint paths in $\mathbf{G}$ *can be found in* $\mathbf{O(mn)}$ *time.*

Proof.

1. Set capacities of edges in $\mathbf{G}$ to $\mathbf{1}$.
2. Run Ford-Fulkerson algorithm.
3. Maximum value of flow is $\mathbf{n}$ and hence run-time is $\mathbf{O(nm)}$.
4. Decompose flow into $\mathbf{k}$ paths ($\mathbf{k} \leq \mathbf{n}$).
   Takes $\mathbf{O(k \times m) = O(km) = O(mn)}$ time. □

Remark

The algorithm also computes a set of edge-disjoint paths realizing this optimal solution.

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Let **G** *be a directed graph. The minimum number of edges whose removal disconnects* **s** *from* **t** *(the minimum-cut between* **s** *and* **t***) is equal to the maximum number of edge-disjoint paths in* **G** *between* **s** *and* **t***.*

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Edge Disjoint Paths in Undirected Graphs

Problem

Given an undirected graph G, find the maximum number of edge disjoint paths in G

Reduction:

1. create directed graph H by adding directed edges (u, v) and (v, u) for each edge uv in G.
2. compute maximum s-t flow in H.

Problem: Both edges (u, v) and (v, u) may have non-zero flow!

Not a Problem! Can assume maximum flow in H is acyclic and hence cannot have non-zero flow on both (u, v) and (v, u). Reduction works. See book for more details.

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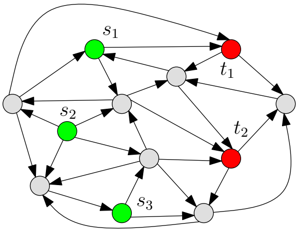
Multiple Sources and Sinks

Input:

1. A directed graph $\mathbf{G}$ with edge capacities $\mathbf{c(e)}$.
2. Source nodes $\mathbf{s_1, s_2, \ldots, s_k}$.
3. Sink nodes $\mathbf{t_1, t_2, \ldots, t_\ell}$.
4. Sources and sinks are *disjoint*.



Maximum Flow: Send as much flow as possible from the sources to the sinks. *Sinks don't care which source they get flow from.*

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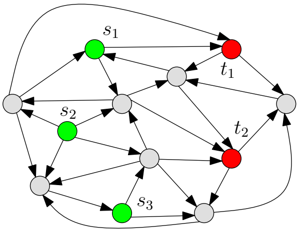

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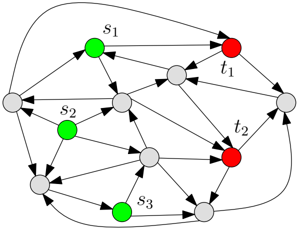

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Multiple Sources and Sinks: Formal Definition

Input:

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2. Source nodes $s_1, s_2, \ldots, s_k$.
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4. Sources and sinks are *disjoint*.

A function $f : E \to \mathbb{R}^{\geq 0}$ is a **flow** if:

1. For each $e \in E$, $f(e) \leq c(e)$, and
2. for each v which is not a source or a sink $f^{in}(v) = f^{out}(v)$.

Goal: max $\sum_{i=1}^{k}(f^{out}(s_i) - f^{in}(s_i))$, that is, flow out of sources.

Reduction to Single-Source Single-Sink

1. Add a **source** node $\mathbf{s}$ and a **sink** node $\mathbf{t}$.
2. Add edges $(\mathbf{s}, \mathbf{s}_1), (\mathbf{s}, \mathbf{s}_2), \ldots, (\mathbf{s}, \mathbf{s}_k)$.
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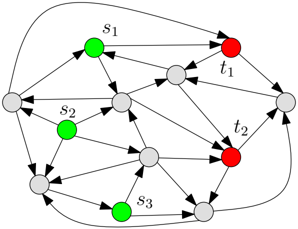

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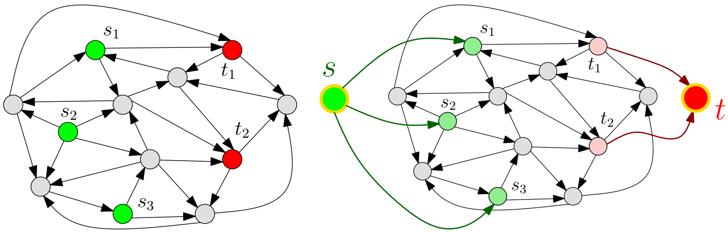
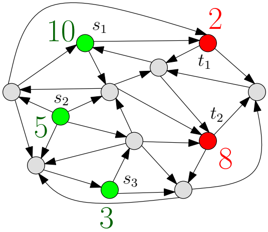


Supplies and Demands

A further generalization:

1. source $\mathbf{s_i}$ has a supply of $\mathbf{S_i \geq 0}$
2. since $\mathbf{t_j}$ has a demand of $\mathbf{D_j \geq 0}$ units

Question: is there a flow from source to sinks such that supplies are not exceeded and demands are met? Formally we have the additional constraints that $\mathbf{f^{out}(s_i) - f^{in}(s_i) \leq S_i}$ for each source $\mathbf{s_i}$ and $\mathbf{f^{in}(t_j) - f^{out}(t_j) \geq D_j}$ for each sink $\mathbf{t_j}$.



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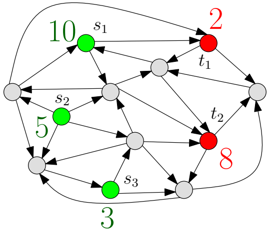
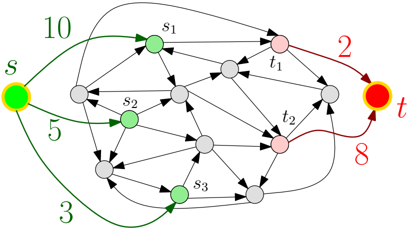


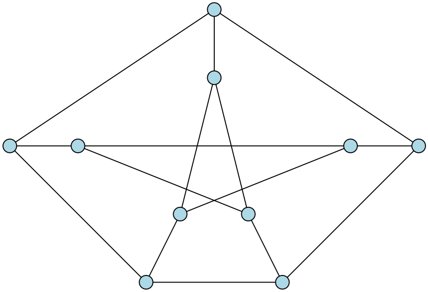


Matching

Problem (Matching)

Input: *Given a (undirected) graph* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$.
Goal: *Find a matching of maximum cardinality.*

1. *A matching is* $\mathbf{M} \subseteq \mathbf{E}$ *such that at most one edge in* $\mathbf{M}$ *is incident on any vertex*



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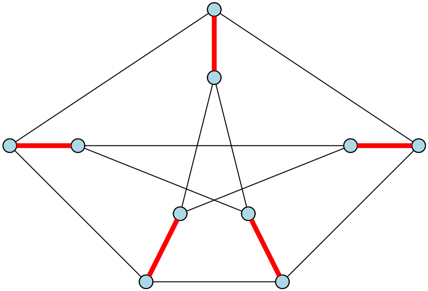

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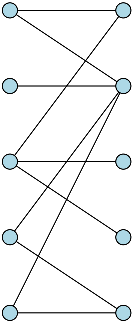

Maximum matching has 4 edges

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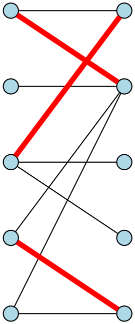

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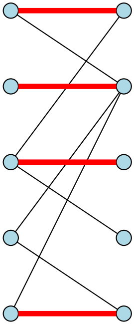

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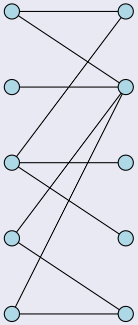
Reduction of bipartite matching to max-flow

Max-Flow Construction

Given graph $\mathbf{G} = (\mathbf{L} \cup \mathbf{R}, \mathbf{E})$ create flow-network $\mathbf{G'} = (\mathbf{V'}, \mathbf{E'})$ as follows:



1. $\mathbf{V'} = \mathbf{L} \cup \mathbf{R} \cup \{\mathbf{s}, \mathbf{t}\}$ where $\mathbf{s}$ and $\mathbf{t}$ are the new source and sink.
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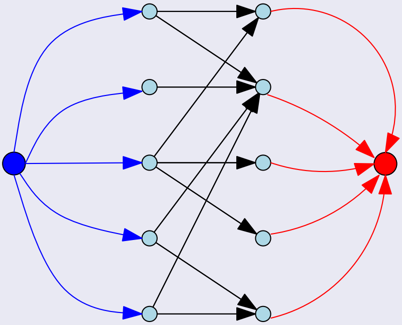

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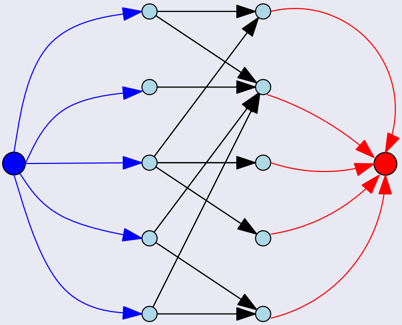

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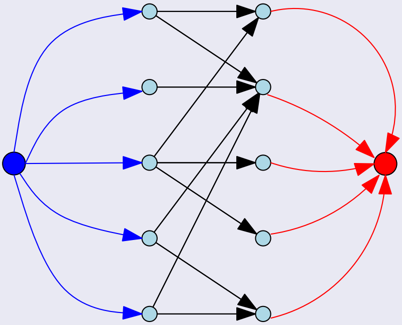

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Correctness: Matching to Flow

Proposition

If $\mathbf{G}$ has a matching of size $\mathbf{k}$ then $\mathbf{G'}$ has a flow of value $\mathbf{k}$.

Proof.

Let $\mathbf{M}$ be matching of size $\mathbf{k}$. Let $\mathbf{M} = \{(\mathbf{u_1}, \mathbf{v_1}), \ldots, (\mathbf{u_k}, \mathbf{v_k})\}$. Consider following flow $\mathbf{f}$ in $\mathbf{G'}$:

1. $\mathbf{f(s, u_i) = 1}$ and $\mathbf{f(v_i, t) = 1}$ for $\mathbf{1 \le i \le k}$
2. $\mathbf{f(u_i, v_i) = 1}$ for $\mathbf{1 \le i \le k}$
3. for all other edges flow is zero.

Verify that $\mathbf{f}$ is a flow of value $\mathbf{k}$ (because $\mathbf{M}$ is a matching). □

Correctness: Matching to Flow

Proposition

If G *has a matching of size* k *then* G' *has a flow of value* k*.*

Proof.

Let M be matching of size k. Let $M = \{(u_1, v_1), \ldots, (u_k, v_k)\}$.
Consider following flow f in G':

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Correctness: Flow to Matching

Proposition

If $\mathbf{G'}$ has a flow of value $\mathbf{k}$ then $\mathbf{G}$ has a matching of size $\mathbf{k}$.

Proof.

Consider flow $\mathbf{f}$ of value $\mathbf{k}$.

1. Can assume $\mathbf{f}$ is integral. Thus each edge has flow $\mathbf{1}$ or $\mathbf{0}$.
2. Consider the set $\mathbf{M}$ of edges from $\mathbf{L}$ to $\mathbf{R}$ that have flow 1.
    1. $\mathbf{M}$ has $\mathbf{k}$ edges because value of flow is equal to the number of non-zero flow edges crossing cut $(\mathbf{L} \cup \{\mathbf{s}\}, \mathbf{R} \cup \{\mathbf{t}\})$
    2. Each vertex has at most one edge in $\mathbf{M}$ incident upon it. Why?

□

Correctness of Reduction

Theorem

The maximum flow value in $\mathbf{G'}$ = *maximum cardinality of matching in* $\mathbf{G}$.

Consequence

Thus, to find maximum cardinality matching in $\mathbf{G}$, we construct $\mathbf{G'}$ and find the maximum flow in $\mathbf{G'}$. Note that the matching itself (not just the value) can be found efficiently from the flow.

Running Time

For graph $\mathbf{G}$ with $\mathbf{n}$ vertices and $\mathbf{m}$ edges $\mathbf{G'}$ has $\mathbf{O(n+m)}$ edges, and $\mathbf{O(n)}$ vertices.

1. Generic Ford-Fulkerson: Running time is $\mathbf{O(mC) = O(nm)}$ since $\mathbf{C = n}$.
2. Capacity scaling: Running time is $\mathbf{O(m^2 \log C) = O(m^2 \log n)}$.

Better running time is known: $\mathbf{O(m\sqrt{n})}$.

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Perfect Matchings

Definition

A matching **M** is said to be **perfect** if every vertex has one edge in **M** incident upon it.

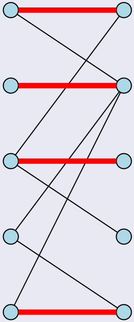

Figure : This graph does not have a perfect matching

Characterizing Perfect Matchings

Problem

When does a bipartite graph have a perfect matching?

1. Clearly $|\mathbf{L}| = |\mathbf{R}|$
2. Are there any necessary and sufficient conditions?

A Necessary Condition

Lemma

If $\mathbf{G} = (\mathbf{L} \cup \mathbf{R}, \mathbf{E})$ *has a perfect matching then for any* $\mathbf{X} \subseteq \mathbf{L}$, $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$, *where* $\mathbf{N}(\mathbf{X})$ *is the set of neighbors of vertices in* $\mathbf{X}$.

Proof.

Since $\mathbf{G}$ has a perfect matching, every vertex of $\mathbf{X}$ is matched to a different neighbor, and so $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$. □

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Hall's Theorem

Theorem (Frobenius-Hall)

Let $\mathbf{G = (L \cup R, E)}$ *be a bipartite graph with* $|\mathbf{L}| = |\mathbf{R}|$*.* $\mathbf{G}$ *has a perfect matching if and only if for every* $\mathbf{X \subseteq L}$*,* $|\mathbf{N(X)}| \geq |\mathbf{X}|$*.*

One direction is the necessary condition.

For the other direction we will show the following:

1. Create flow network $\mathbf{G'}$ from $\mathbf{G}$.
2. If $|\mathbf{N(X)}| \geq |\mathbf{X}|$ for all $\mathbf{X}$, show that minimum $\mathbf{s}$-$\mathbf{t}$ cut in $\mathbf{G'}$ is of capacity $\mathbf{n} = |\mathbf{L}| = |\mathbf{R}|$.
3. Implies that $\mathbf{G}$ has a perfect matching.

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For the other direction we will show the following:

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2. If $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$ for all $\mathbf{X}$, show that minimum $\mathbf{s}$-$\mathbf{t}$ cut in $\mathbf{G}'$ is of capacity $\mathbf{n} = |\mathbf{L}| = |\mathbf{R}|$.
3. Implies that $\mathbf{G}$ has a perfect matching.

Proof of Sufficiency

Assume $|N(X)| \geq |X|$ for any $X \subseteq L$. Then show that min s-t cut in G' is of capacity at least n.

Let (A, B) be an *arbitrary* s-t cut in G'

1. Let $X = A \cap L$ and $Y = A \cap R$.
2. Cut capacity is at least $(|L| - |X|) + |Y| + |N(X) \setminus Y|$

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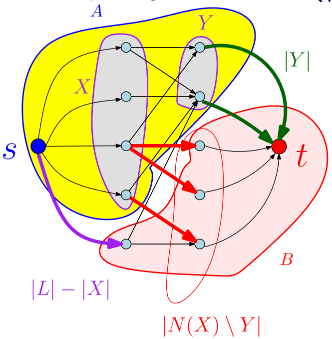

Because there are...

1. $|\mathbf{L}| - |\mathbf{X}|$ edges from **s** to $\mathbf{L} \cap \mathbf{B}$.
2. $|\mathbf{Y}|$ edges from **Y** to **t**.
3. there are at least $|\mathbf{N(X)} \setminus \mathbf{Y}|$ edges from **X** to vertices on the right side that are not in **Y**.

Proof of Sufficiency

Continued...

1. By the above, cut capacity is at least
$$\alpha = (|\mathbf{L}| - |\mathbf{X}|) + |\mathbf{Y}| + |\mathbf{N(X)} \setminus \mathbf{Y}|.$$
2. $|\mathbf{N(X)} \setminus \mathbf{Y}| \geq |\mathbf{N(X)}| - |\mathbf{Y}|$.
(This holds for any two sets.)
3. By assumption $|\mathbf{N(X)}| \geq |\mathbf{X}|$ and hence
$$|\mathbf{N(X)} \setminus \mathbf{Y}| \geq |\mathbf{N(X)}| - |\mathbf{Y}| \geq |\mathbf{X}| - |\mathbf{Y}|.$$
4. Cut capacity is therefore at least
$$\alpha = (|\mathbf{L}| - |\mathbf{X}|) + |\mathbf{Y}| + |\mathbf{N(X)} \setminus \mathbf{Y}| \\ \geq |\mathbf{L}| - |\mathbf{X}| + |\mathbf{Y}| + |\mathbf{X}| - |\mathbf{Y}| \geq |\mathbf{L}| = \mathbf{n}.$$
5. Any **s**-**t** cut capacity is at least **n** $\implies$ max flow at least **n** units $\implies$ perfect matching. **QED**

Hall's Theorem: Generalization

Theorem (Frobenius-Hall)

Let $\mathbf{G} = (\mathbf{L} \cup \mathbf{R}, \mathbf{E})$ *be a bipartite graph with* $|\mathbf{L}| \leq |\mathbf{R}|$*.* $\mathbf{G}$ *has a matching that matches all nodes in* $\mathbf{L}$ *if and only if for every* $\mathbf{X} \subseteq \mathbf{L}$*,* $|\mathbf{N}(\mathbf{X})| \geq |\mathbf{X}|$*.*

Proof is essentially the same as the previous one.

Assigning jobs to people

1. $\mathbf{n}$ jobs, $\mathbf{n/2}$ people
2. For each job: a set of people who can do that job.
3. Each person $\mathbf{j}$ has to do exactly two jobs.
4. Goal: find an assignment of 2 jobs to each person, such that all jobs are assigned.

Solution: Build bipartite graph, compute maximum matching, remove it, compute another maximum matching. Both matchings together form a valid solution if it exists. This algorithm is

(A) Correct.

(B) Incorrect.

Application: Assigning jobs to people

1. **n** jobs or tasks
2. **m** people
3. for each job a set of people who can do that job
4. for each person **j** a limit on number of jobs $\mathbf{k_j}$
5. Goal: find an assignment of jobs to people so that all jobs are assigned and no person is overloaded

Reduce to max-flow similar to matching.

Arises in many settings. Using *minimum-cost flows* can also handle the case when assigning a job **i** to person **j** costs $\mathbf{c_{ij}}$ and goal is assign all jobs but minimize cost of assignment.

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Reduction to Maximum Flow

1. Create directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ as follows
   1. $\mathbf{V} = \{\mathbf{s}, \mathbf{t}\} \cup \mathbf{L} \cup \mathbf{R}$: $\mathbf{L}$ set of $\mathbf{n}$ jobs, $\mathbf{R}$ set of $\mathbf{m}$ people
   2. add edges $(\mathbf{s}, \mathbf{i})$ for each job $\mathbf{i} \in \mathbf{L}$, capacity $\mathbf{1}$
   3. add edges $(\mathbf{j}, \mathbf{t})$ for each person $\mathbf{j} \in \mathbf{R}$, capacity $\mathbf{k_j}$
   4. if job $\mathbf{i}$ can be done by person $\mathbf{j}$ add an edge $(\mathbf{i}, \mathbf{j})$, capacity $\mathbf{1}$
2. Compute max $\mathbf{s}$-$\mathbf{t}$ flow. There is an assignment if and only if flow value is $\mathbf{n}$.

Matchings in General Graphs

Matchings in general graphs more complicated.

There is a polynomial time algorithm to compute a maximum matching in a general graph. Best known running time is $\mathbf{O(m\sqrt{n})}$.

Notes

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