Breadth First Search, Dijkstra's Algorithm for Shortest Paths

Lecture 3

September 2, 2014

Part I

Breadth First Search

Breadth First Search (BFS)

Overview

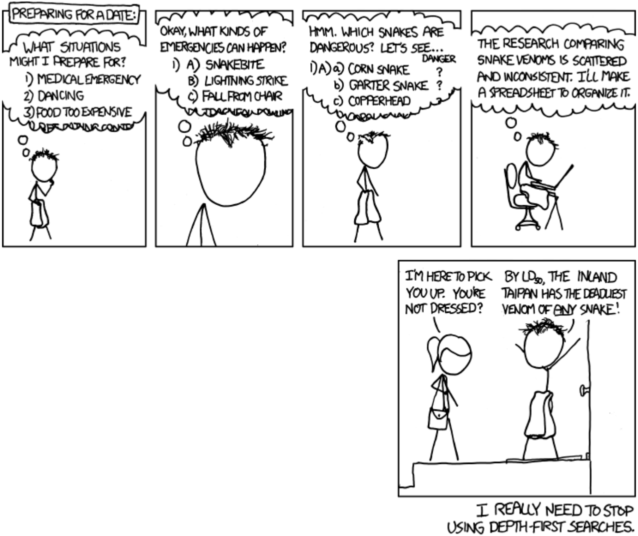
(A) **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex **s** (the start vertex).

As such...

1. **DFS** good for exploring graph structure
2. **BFS** good for exploring *distances*

xkcd take on DFS



Queue Data Structure

Queues

A **queue** is a list of elements which supports the operations:

1. **enqueue**: Adds an element to the end of the list
2. **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.

BFS Algorithm

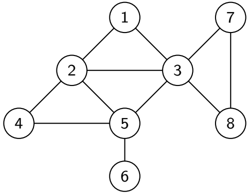
Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

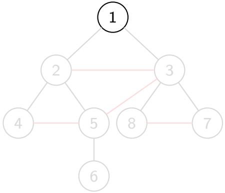
```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enq(v)
```

Proposition

$BFS(s)$ *runs in* $O(n + m)$ *time.*

BFS: An Example in Undirected Graphs





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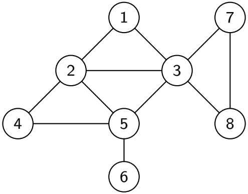
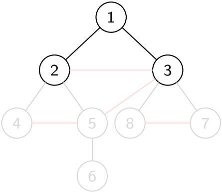
BFS tree is the set of black edges.

BFS: An Example in Undirected Graphs





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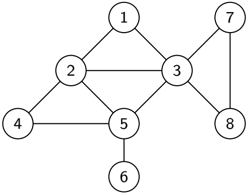

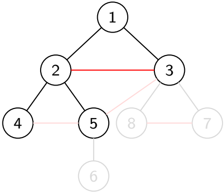

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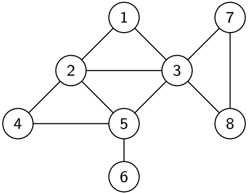

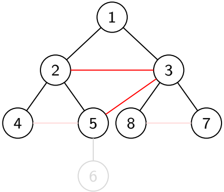

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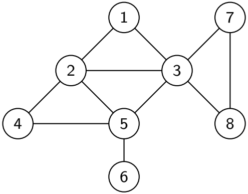

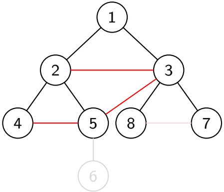

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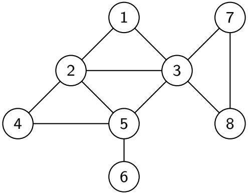

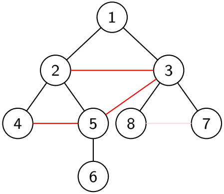

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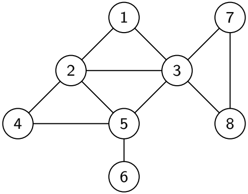

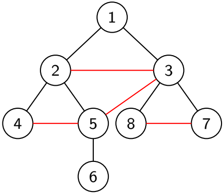

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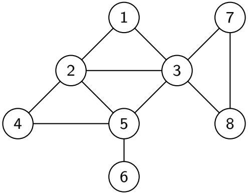

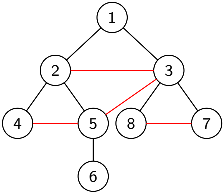

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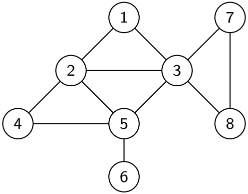

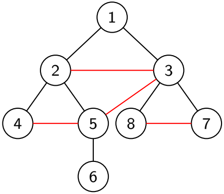

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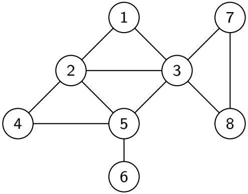

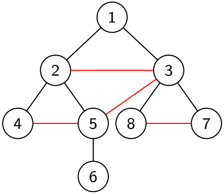

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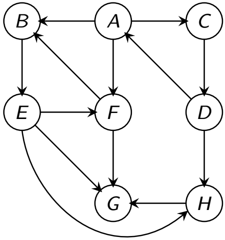
BFS: An Example in Directed Graphs



BFS with Distance

```
BFS(s)
    Mark all vertices as unvisited
and for each v set dist(v) = ∞
    Initialize search tree T to be empty
    Mark vertex s as visited and set dist(s) = 0
    set Q to be the empty queue
    enq(s)
    while Q is nonempty do
        u = deq(Q)
        for each vertex v ∈ Adj(u) do
            if v is not visited do
                add edge (u, v) to T
                Mark v as visited, enq(v)
                and set dist(v) = dist(u) + 1
```

Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of **BFS**$(\mathbf{s})$

(A) *The search tree contains exactly the set of vertices in the connected component of* $\mathbf{s}$.

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* $\mathbf{u}$ *is visited before* $\mathbf{v}$.

(C) *For every vertex* $\mathbf{u}$, $\mathrm{dist}(\mathbf{u})$ *is indeed the length of shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$.

(D) *If* $\mathbf{u}, \mathbf{v}$ *are in connected component of* $\mathbf{s}$ *and* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is an edge of* $\mathbf{G}$, *then either* $\mathbf{e}$ *is an edge in the search tree, or* $|\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v})| \leq 1$.

Proof.

Exercise. □

Question: For directed graphs...

If the edge $(\mathbf{u}, \mathbf{v})$ is in the graph G.
Do **BFS** ($\mathbf{s}$), such that both $\mathbf{u}$ and $\mathbf{v}$ are reachable from $\mathbf{s}$.
Then, we must have that

(A) $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$

(B) $\mathrm{dist}(\mathbf{u}) > \mathrm{dist}(\mathbf{v})$

(C) $\mathrm{dist}(\mathbf{u}) \leq 1 + \mathrm{dist}(\mathbf{v})$

(D) $\mathrm{dist}(\mathbf{v}) \leq 1 + \mathrm{dist}(\mathbf{u})$

(E) None of the above.

(F) All of the above.

Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of **BFS**(**s**)*:*

(A) *The search tree contains exactly the set of vertices reachable from* **s**

(B) *If* $\mathrm{dist}(\mathbf{u}) < \mathrm{dist}(\mathbf{v})$ *then* **u** *is visited before* **v**

(C) *For every vertex* **u**, $\mathrm{dist}(\mathbf{u})$ *is indeed the length of shortest path from* **s** *to* **u**

(D) *If* **u** *is reachable from* **s** *and* $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ *is an edge of* **G**, *then either* **e** *is an edge in the search tree, or* $\mathrm{dist}(\mathbf{v}) - \mathrm{dist}(\mathbf{u}) \leq 1$. *Not necessarily the case that* $\mathrm{dist}(\mathbf{u}) - \mathrm{dist}(\mathbf{v}) \leq 1$.

Proof.

Exercise. □

BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
            initialize L_{i+1} to be an empty list
            for each u in L_i do
                for each edge (u, v) ∈ Adj(u) do
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
```

Running time: $O(n + m)$

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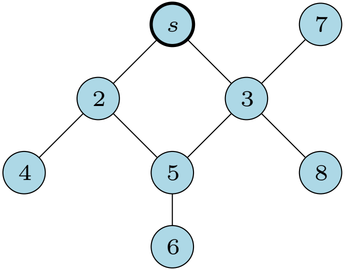
Example



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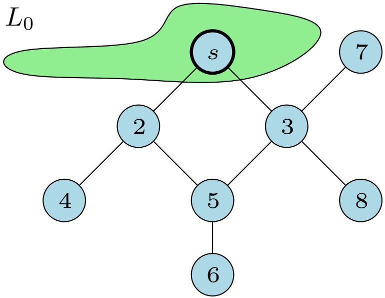

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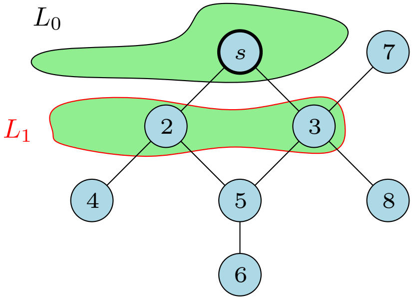

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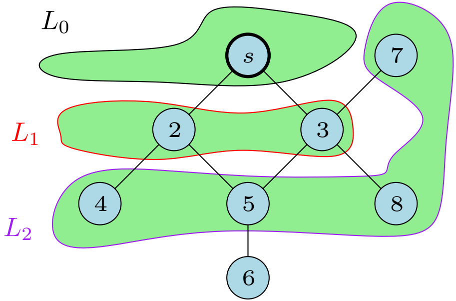

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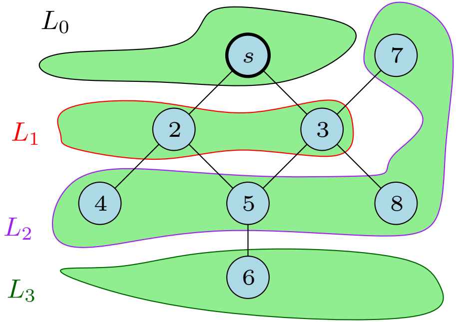


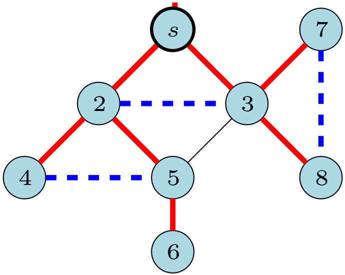
BFS with Layers: Properties

Proposition

The following properties hold on termination of **BFSLayers**($\mathbf{s}$).

1. **BFSLayers**($\mathbf{s}$) *outputs a* **BFS** *tree*
2. $\mathbf{L_i}$ *is the set of vertices at distance exactly* $\mathbf{i}$ *from* $\mathbf{s}$
3. *If* $\mathbf{G}$ *is undirected, each edge* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *is one of three types:*
   1. **tree** *edge between two* consecutive *layers*
   2. *non-tree* **forward**/**backward** *edge between two consecutive layers*
   3. *non-tree* **cross-edge** *with both* $\mathbf{u}, \mathbf{v}$ *in same layer*
   4. $\implies$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

Example



BFS with Layers: Properties

For directed graphs

Proposition

The following properties hold on termination of **BFSLayers**$(\mathbf{s})$*, if* $\mathbf{G}$ *is directed.*

For each edge $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ *is one of four types:*

1. *a* **tree** *edge between consecutive layers,* $\mathbf{u} \in \mathbf{L_i}, \mathbf{v} \in \mathbf{L_{i+1}}$ *for some* $\mathbf{i} \geq \mathbf{0}$
2. *a non-tree* **forward** *edge between consecutive layers*
3. *a non-tree* **backward** *edge*
4. *a* **cross-edge** *with both* $\mathbf{u}, \mathbf{v}$ *in same layer*

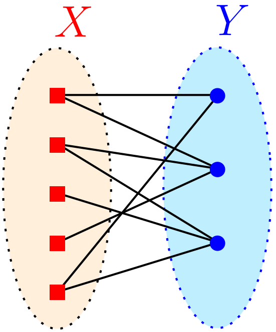
Part II

Bipartite Graphs and an application of BFS

Bipartite Graphs

Definition (Bipartite Graph)

Undirected graph $\mathbf{G = (V, E)}$ is a **bipartite graph** if $\mathbf{V}$ can be partitioned into $\mathbf{X}$ and $\mathbf{Y}$ s.t. all edges in $\mathbf{E}$ are between $\mathbf{X}$ and $\mathbf{Y}$.



Bipartite Graph Characterization

Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree T at some node r. Let L_i be all nodes at level i, that is, L_i is all nodes at distance i from root r. Now define X to be all nodes at even levels and Y to be all nodes at odd level. Only edges in T are between levels. □

Proposition

An odd length cycle is not bipartite.

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Odd Cycles are not Bipartite

Proposition

An odd length cycle is not bipartite.

Proof.

Let $C = u_1, u_2, \ldots, u_{2k+1}, u_1$ be an odd cycle. Suppose C is a bipartite graph and let X, Y be the partition. Without loss of generality $u_1 \in X$. Implies $u_2 \in Y$. Implies $u_3 \in X$. Inductively, $u_i \in X$ if i is odd $u_i \in Y$ if i is even. But $\{u_1, u_{2k+1}\}$ is an edge and both belong to X! □

Subgraphs

Definition

Given a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ a **subgraph** of $\mathbf{G}$ is another graph $\mathbf{H} = (\mathbf{V}', \mathbf{E}')$ where $\mathbf{V}' \subseteq \mathbf{V}$ and $\mathbf{E}' \subseteq \mathbf{E}$.

Proposition

If $\mathbf{G}$ is bipartite then any subgraph $\mathbf{H}$ of $\mathbf{G}$ is also bipartite.

Proposition

A graph $\mathbf{G}$ is not bipartite if $\mathbf{G}$ has an odd cycle $\mathbf{C}$ as a subgraph.

Proof.

If $\mathbf{G}$ is bipartite then since $\mathbf{C}$ is a subgraph, $\mathbf{C}$ is also bipartite (by above proposition). However, $\mathbf{C}$ is not bipartite! □

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Bipartite Graph Characterization

Theorem

A graph $\mathbf{G}$ *is bipartite if and only if it has no odd length cycle as subgraph.*

Proof.

Only If: $\mathbf{G}$ has an odd cycle implies $\mathbf{G}$ is not bipartite.

If: $\mathbf{G}$ has no odd length cycle. Assume without loss of generality that $\mathbf{G}$ is connected.

1. Pick $\mathbf{u}$ arbitrarily and do $\mathbf{BFS}(\mathbf{u})$
2. $\mathbf{X} = \cup_{\mathbf{i} \text{ is even}} \mathbf{L_i}$ and $\mathbf{Y} = \cup_{\mathbf{i} \text{ is odd}} \mathbf{L_i}$
3. **Claim:** $\mathbf{X}$ and $\mathbf{Y}$ is a valid partition if $\mathbf{G}$ has no odd length cycle.

□

Bipartite Graph Characterization

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A graph **G** *is bipartite if and only if it has no odd length cycle as subgraph.*

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If: **G** has no odd length cycle. Assume without loss of generality that **G** is connected.

1. Pick **u** arbitrarily and do **BFS**(**u**)
2. $\mathbf{X} = \cup_{\mathbf{i} \text{ is even}} \mathbf{L_i}$ and $\mathbf{Y} = \cup_{\mathbf{i} \text{ is odd}} \mathbf{L_i}$
3. **Claim:** **X** and **Y** is a valid partition if **G** has no odd length cycle.

□

Proof of Claim

Claim

In **BFS**$(\mathbf{u})$ *if* $\mathbf{a}, \mathbf{b} \in \mathbf{L_i}$ *and* $(\mathbf{a}, \mathbf{b})$ *is an edge then there is an odd length cycle containing* $(\mathbf{a}, \mathbf{b})$*.*

Proof.

Let $\mathbf{v}$ be least common ancestor of $\mathbf{a}, \mathbf{b}$ in **BFS** tree $\mathbf{T}$.
$\mathbf{v}$ is in some level $\mathbf{j} < \mathbf{i}$ (could be $\mathbf{u}$ itself).
Path from $\mathbf{v} \rightsquigarrow \mathbf{a}$ in $\mathbf{T}$ is of length $\mathbf{j} - \mathbf{i}$.
Path from $\mathbf{v} \rightsquigarrow \mathbf{b}$ in $\mathbf{T}$ is of length $\mathbf{j} - \mathbf{i}$.
These two paths plus $(\mathbf{a}, \mathbf{b})$ forms an odd cycle of length $2(\mathbf{j} - \mathbf{i}) + 1$. □

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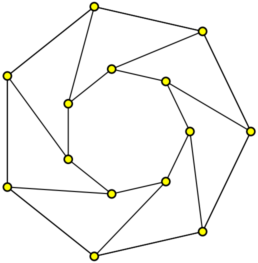
These two paths plus $(\mathbf{a}, \mathbf{b})$ forms an odd cycle of length $\mathbf{2(j - i) + 1}$. □

Proof of Claim: Figure

Question: Is the following graph bipartite?



(A) Yes.

(B) No.

(C) IDK.

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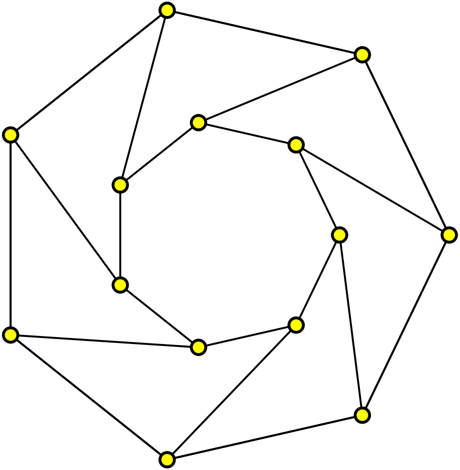

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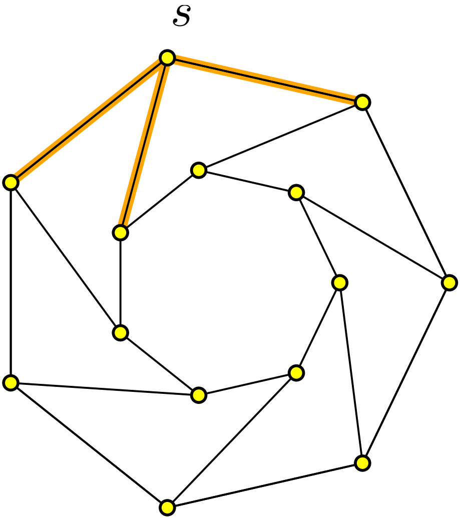

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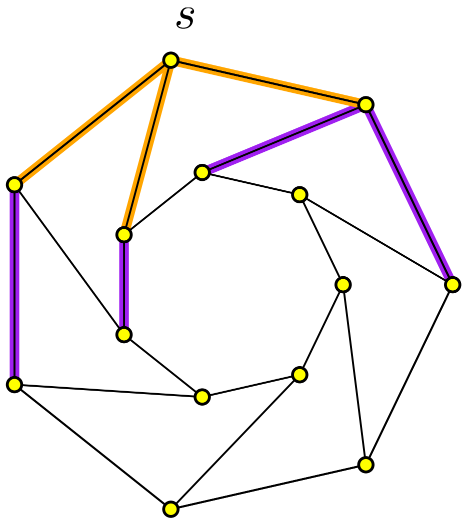

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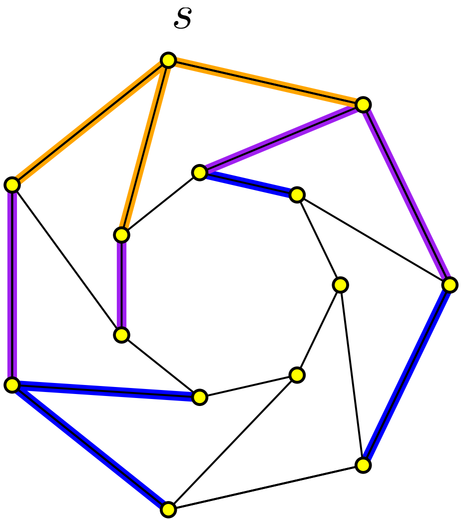

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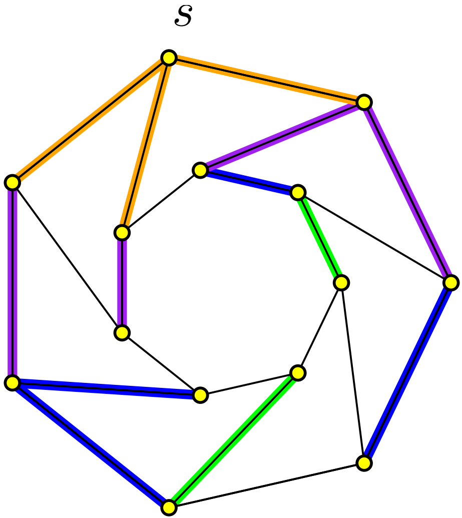

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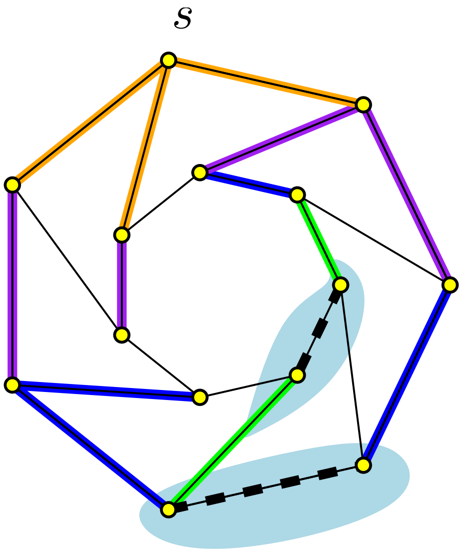

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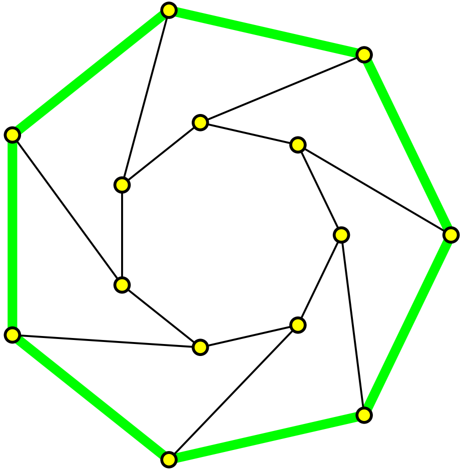

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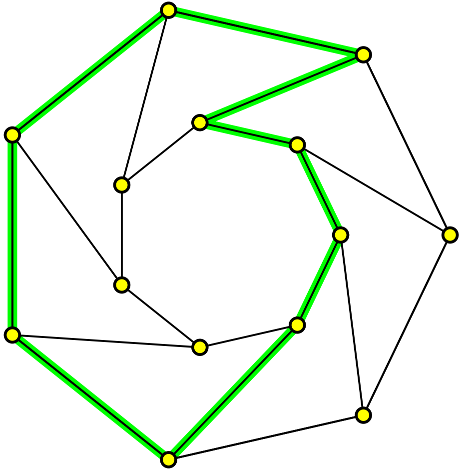

Another tidbit

Corollary

There is an $\mathbf{O(n+m)}$ *time algorithm to check if* $\mathbf{G}$ *is bipartite and output an odd cycle if it is not.*

Part III

Shortest Paths and Dijkstra's Algorithm

Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes s, t find shortest path from s to t.
2. Given node s find shortest path from s to all other nodes.
3. Find shortest paths for all pairs of nodes.

Many applications!

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Single-Source Shortest Paths:

Non-Negative Edge Lengths

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1. Restrict attention to directed graphs
2. Undirected graph problem can be reduced to directed graph problem - how?
   1. Given undirected graph $\mathbf{G}$, create a new directed graph $\mathbf{G}'$ by replacing each edge $\{\mathbf{u}, \mathbf{v}\}$ in $\mathbf{G}$ by $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{v}, \mathbf{u})$ in $\mathbf{G}'$.
   2. set $\ell(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}, \mathbf{u}) = \ell(\{\mathbf{u}, \mathbf{v}\})$
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Single-Source Shortest Paths via BFS

Special case: All edge lengths are $\mathbf{1}$.

1. Run **BFS**($\mathbf{s}$) to get shortest path distances from s to all other nodes.
2. $\mathbf{O(m+n)}$ time algorithm.

Special case: Suppose $\ell(\mathbf{e})$ is an integer for all $\mathbf{e}$?
Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(\mathbf{e}) - \mathbf{1}$ dummy nodes on $\mathbf{e}$

Let $\mathbf{L} = \max_{\mathbf{e}} \ell(\mathbf{e})$. New graph has $\mathbf{O(mL)}$ edges and $\mathbf{O(mL+n)}$ nodes. **BFS** takes $\mathbf{O(mL+n)}$ time. Not efficient if $\mathbf{L}$ is large.

Single-Source Shortest Paths via BFS

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Towards an algorithm

Why does **BFS** work?

BFS(s) explores nodes in increasing distance from s

Lemma

Let G *be a directed graph with non-negative edge lengths. Let* $\mathrm{dist}(s, v)$ *denote the shortest path length from* s *to* v*. If* $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ *is a shortest path from* s *to* v_k *then for* $1 \leq i < k$*:*

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2. $\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$.

Proof.

Suppose not. Then for some $i < k$ there is a path P' from s to v_i of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then P' concatenated with $v_i \to v_{i+1} \ldots \to v_k$ contains a strictly shorter

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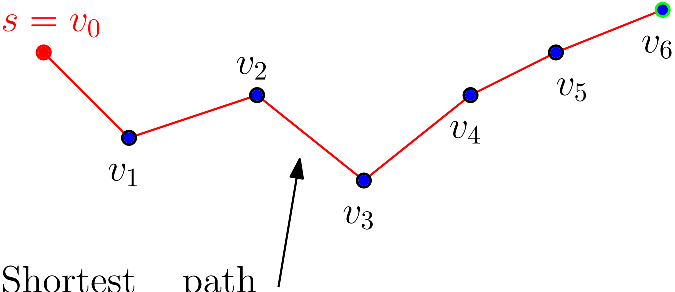
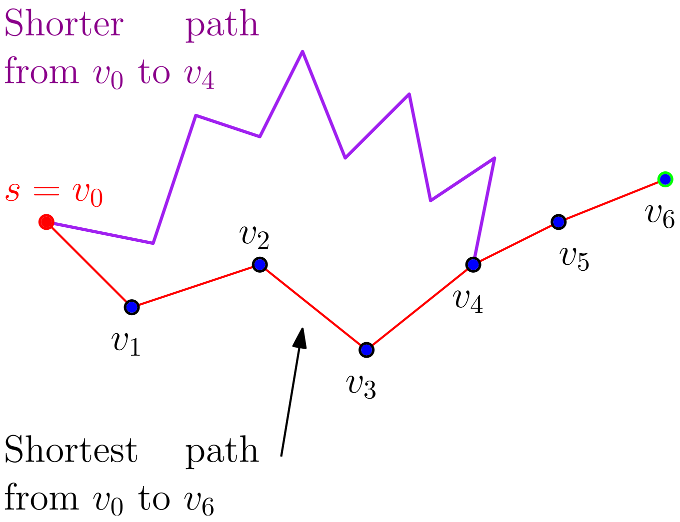
A proof by picture



Shortest path from v_0 to v_6

A proof by picture



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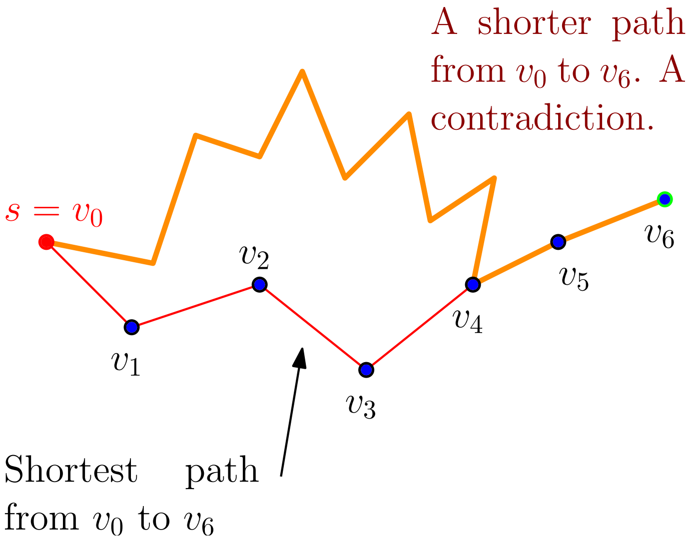

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

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Initialize for each node v, dist(s, v) = ∞
Initialize S = ∅,
for i = 1 to |V| do
    (* Invariant:  S contains the i − 1 closest nodes to s *)
    Among nodes in V \ S, find the node v that is the
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2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V - S}$.

What do we know about the $\mathbf{i}$th closest node?

Claim

Let $\mathbf{P}$ be a shortest path from $\mathbf{s}$ to $\mathbf{v}$ where $\mathbf{v}$ is the $\mathbf{i}$th closest node. Then, all intermediate nodes in $\mathbf{P}$ belong to $\mathbf{S}$.

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If $\mathbf{P}$ had an intermediate node $\mathbf{u}$ not in $\mathbf{S}$ then $\mathbf{u}$ will be closer to $\mathbf{s}$ than $\mathbf{v}$. Implies $\mathbf{v}$ is not the $\mathbf{i}$th closest node to $\mathbf{s}$ - recall that $\mathbf{S}$ already has the $\mathbf{i-1}$ closest nodes. □

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Proof.

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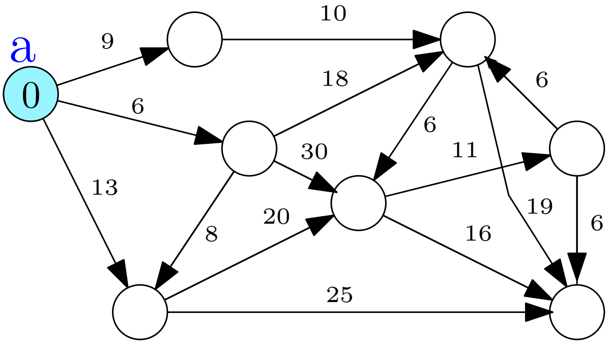
Finding the ith closest node repeatedly

An example



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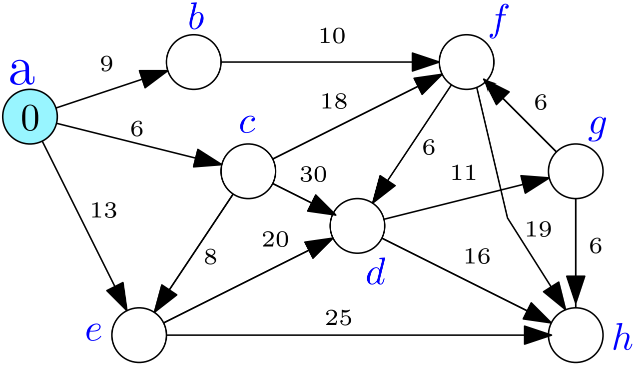

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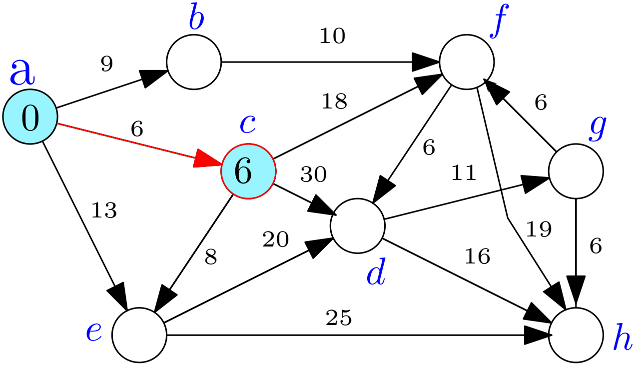

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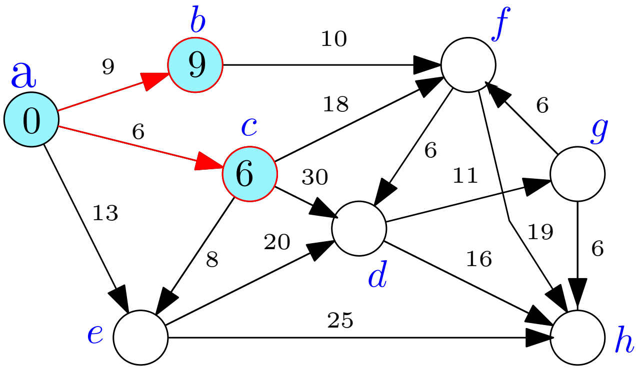

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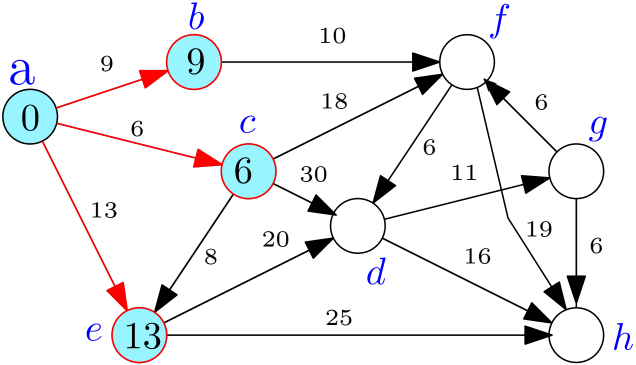

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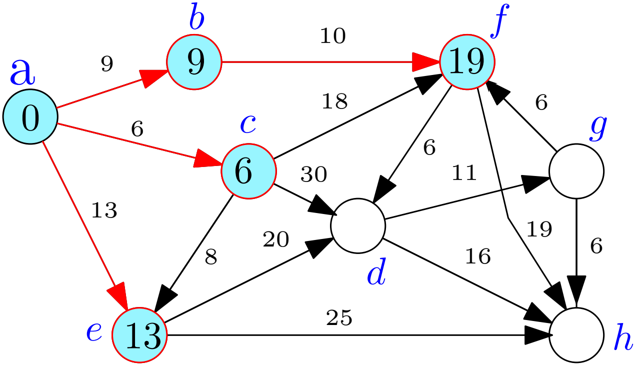

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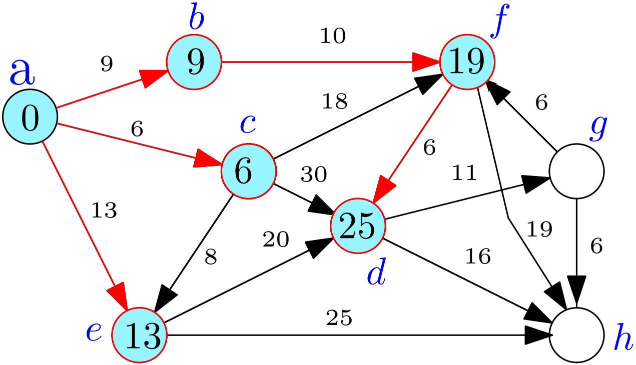

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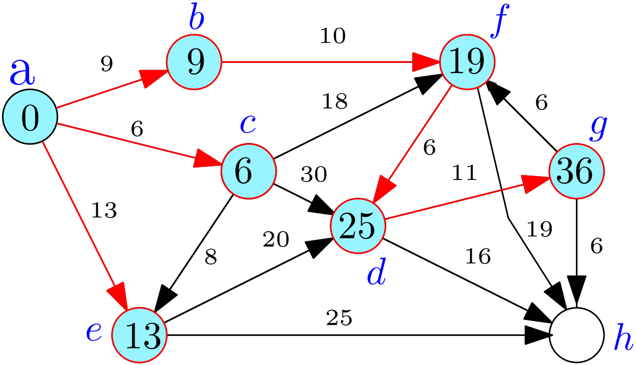

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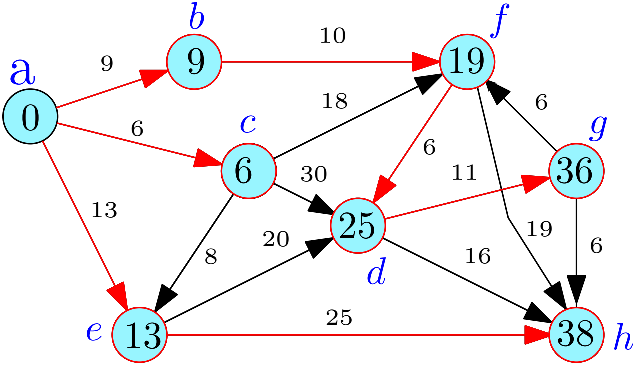

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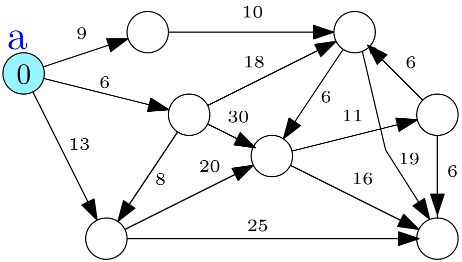

Corollary

The **i***th closest node is adjacent to* **S***.*

Finding the ith closest node

1. $\mathbf{S}$ contains the $\mathbf{i-1}$ closest nodes to $\mathbf{s}$
2. Want to find the $\mathbf{i}$th closest node from $\mathbf{V-S}$.

1. For each $\mathbf{u \in V - S}$ let $\mathbf{P(s, u, S)}$ be a shortest path from $\mathbf{s}$ to $\mathbf{u}$ using only nodes in $\mathbf{S}$ as intermediate vertices.
2. Let $\mathbf{d'(s, u)}$ be the length of $\mathbf{P(s, u, S)}$

Observations: for each $\mathbf{u \in V - S}$,

1. $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \leq \mathbf{d'(s, u)}$ since we are constraining the paths
2. $\mathbf{d'(s, u)} = \mathbf{min}_{\mathbf{a \in S}}(\mathrm{dist}(\mathbf{s}, \mathbf{a}) + \ell(\mathbf{a}, \mathbf{u}))$ - Why?

Lemma

If $\mathbf{v}$ is the $\mathbf{i}$th closest node to $\mathbf{s}$, then $\mathbf{d'(s, v)} = \mathrm{dist}(\mathbf{s}, \mathbf{v})$.

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Observations: for each $\mathbf{u \in V - S}$,

1. $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \leq \mathbf{d'(s, u)}$ since we are constraining the paths
2. $\mathbf{d'(s, u)} = \mathbf{min}_{\mathbf{a \in S}}(\mathrm{dist}(\mathbf{s}, \mathbf{a}) + \ell(\mathbf{a}, \mathbf{u}))$ - Why?

Lemma

If $\mathbf{v}$ is the $\mathbf{i}$th closest node to $\mathbf{s}$, then $\mathbf{d'(s, v)} = \mathrm{dist}(\mathbf{s}, \mathbf{v})$.

Finding the ith closest node

Lemma

Given:

1. S: *Set of* $i - 1$ *closest nodes to* s.
2. $d'(s, u) = \min_{x \in S}(\mathrm{dist}(s, x) + \ell(x, u))$

If v *is an* i*th closest node to* s*, then* $d'(s, v) = \mathrm{dist}(s, v)$.

Proof.

Let v be the ith closest node to s. Then there is a shortest path P from s to v that contains only nodes in S as intermediate nodes (see previous claim). Therefore $d'(s, v) = \mathrm{dist}(s, v)$. □

Finding the ith closest node

Lemma

If $\mathbf{v}$ is an $\mathbf{i}$th closest node to $\mathbf{s}$, then $\mathbf{d'(s,v)} = \mathrm{dist}(\mathbf{s},\mathbf{v})$.

Corollary

The $\mathbf{i}$th closest node to $\mathbf{s}$ is the node $\mathbf{v} \in \mathbf{V} - \mathbf{S}$ such that $\mathbf{d'(s,v)} = \mathbf{min_{u \in V-S}\, d'(s,u)}$.

Proof.

For every node $\mathbf{u} \in \mathbf{V} - \mathbf{S}$, $\mathrm{dist}(\mathbf{s},\mathbf{u}) \leq \mathbf{d'(s,u)}$ and for the $\mathbf{i}$th closest node $\mathbf{v}$, $\mathrm{dist}(\mathbf{s},\mathbf{v}) = \mathbf{d'(s,v)}$. Moreover, $\mathrm{dist}(\mathbf{s},\mathbf{u}) \geq \mathrm{dist}(\mathbf{s},\mathbf{v})$ for each $\mathbf{u} \in \mathbf{V} - \mathbf{S}$. □

Algorithm

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Initialize for each node v:  dist(s, v) = ∞
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     using only S as intermediate nodes*)
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    S = S ∪ {v}
    for each node u in V \ S do
        d'(s, u) ⇐ min_{a∈S} (dist(s, a) + ℓ(a, u))
```

Correctness: By induction on i using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

1. n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in S; $O(m + n)$ time/iteration.

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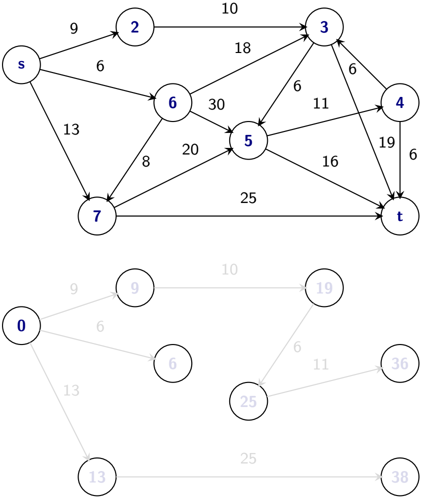
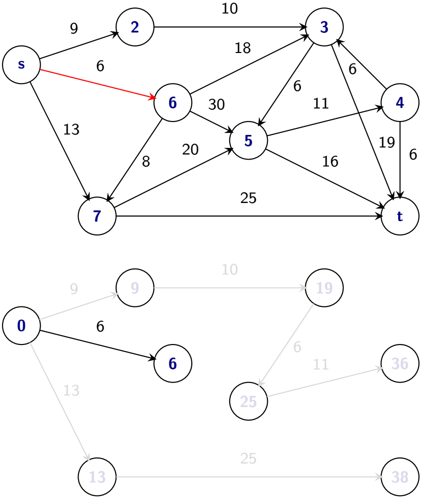
Example



Example



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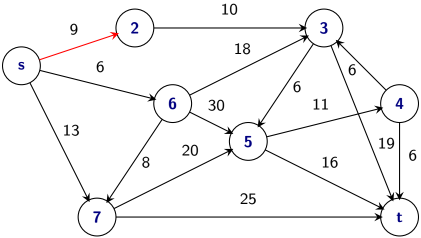

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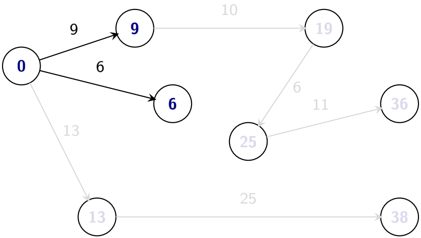

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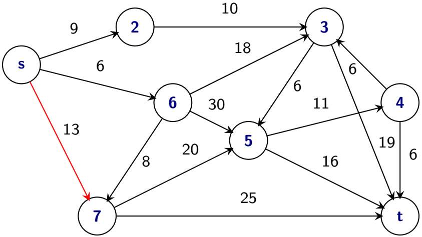

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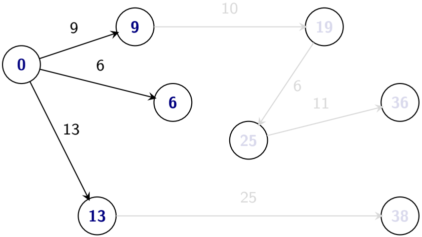

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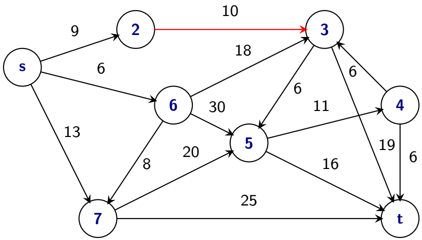

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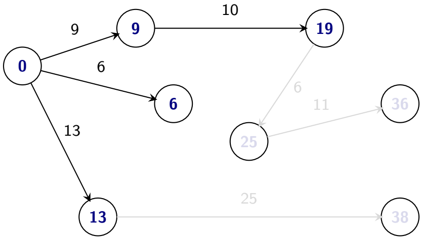

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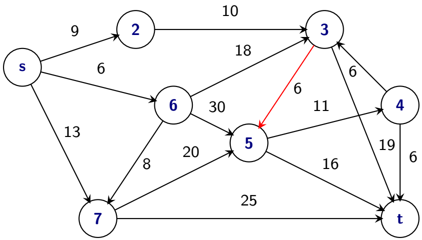

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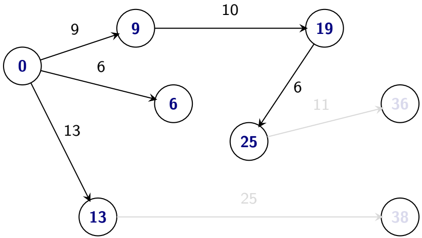

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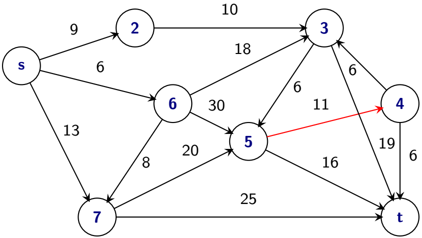

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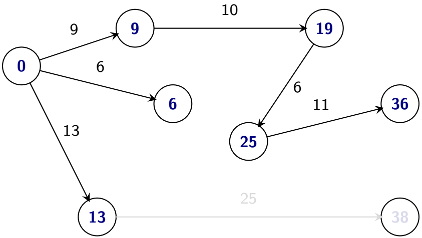

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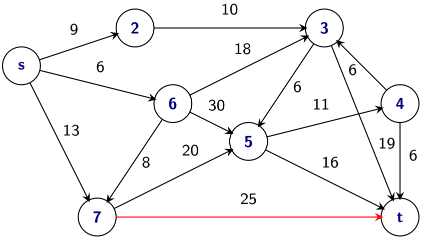
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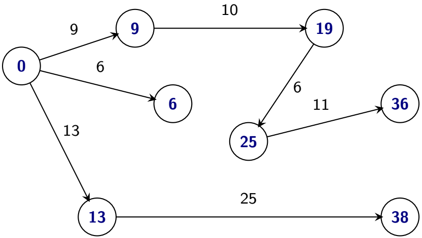
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Improved Algorithm

1. Main work is to compute the $\mathbf{d'(s,u)}$ values in each iteration
2. $\mathbf{d'(s,u)}$ changes from iteration $\mathbf{i}$ to $\mathbf{i+1}$ only because of the node $\mathbf{v}$ that is added to $\mathbf{S}$ in iteration $\mathbf{i}$.

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Running time: $O(m + n^2)$ time.

1. $\mathbf{n}$ outer iterations and in each iteration following steps
2. updating $\mathbf{d'(s,u)}$ after $\mathbf{v}$ added takes $O(\deg(v))$ time so total

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1. n outer iterations and in each iteration following steps
2. updating $d'(s, u)$ after v added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters S only once
3. Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

1. eliminate $\mathbf{d'(s,u)}$ and let $\mathrm{dist}(\mathbf{s},\mathbf{u})$ maintain it
2. update **dist** values after adding **v** by scanning edges out of **v**

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Initialize for each node v, dist(s, v) = ∞
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Priority Queues to maintain **dist** values for faster running time

1. Using heaps and standard priority queues: $O((m+n)\log n)$
2. Using Fibonacci heaps: $O(m + n\log n)$.

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Priority Queues

Data structure to store a set **S** of **n** elements where each element $\mathbf{v} \in \mathbf{S}$ has an associated real/integer key $\mathbf{k(v)}$ such that the following operations:

1. **makePQ**: create an empty queue.
2. **findMin**: find the minimum key in **S**.
3. **extractMin**: Remove $\mathbf{v} \in \mathbf{S}$ with smallest key and return it.
4. **insert**($\mathbf{v}, \mathbf{k(v)}$): Add new element **v** with key $\mathbf{k(v)}$ to **S**.
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6. **decreaseKey**($\mathbf{v}, \mathbf{k'(v)}$): *decrease* key of **v** from $\mathbf{k(v)}$ (current key) to $\mathbf{k'(v)}$ (new key). Assumption: $\mathbf{k'(v)} \leq \mathbf{k(v)}$.
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All operations can be performed in $\mathbf{O(\log n)}$ time.
decreaseKey is implemented via **delete** and **insert**.

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Dijkstra's Algorithm using Priority Queues

$\mathbf{Q} \Leftarrow$ **makePQ**()
insert($\mathbf{Q}$, $(\mathbf{s}, 0)$)
for each node $\mathbf{u} \neq \mathbf{s}$ **do**
    insert($\mathbf{Q}$, $(\mathbf{u}, \infty)$)
$\mathbf{S} \Leftarrow \emptyset$
for $\mathbf{i} = 1$ to $|\mathbf{V}|$ **do**
    $(\mathbf{v}, \mathrm{dist}(\mathbf{s}, \mathbf{v})) =$ **extractMin(Q)**
    $\mathbf{S} = \mathbf{S} \cup \{\mathbf{v}\}$
    for each $\mathbf{u}$ in $\mathrm{Adj}(\mathbf{v})$ **do**
        decreaseKey$\Big(\mathbf{Q}, \big(\mathbf{u}, \min\big(\mathrm{dist}(\mathbf{s}, \mathbf{u}),\ \mathrm{dist}(\mathbf{s}, \mathbf{v}) + \ell(\mathbf{v}, \mathbf{u})\big)\big)\Big)$.

Priority Queue operations:

1. $O(n)$ **insert** operations
2. $O(n)$ **extractMin** operations
3. $O(m)$ **decreaseKey** operations

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

1. All operations can be done in $\mathbf{O(\log n)}$ time

Dijkstra's algorithm can be implemented in $\mathbf{O((n + m) \log n)}$ time.

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Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **insert**, **delete**, **meld** in $O(\log n)$ time
2. **decreaseKey** in $O(1)$ *amortized* time: ℓ **decreaseKey** operations for $\ell \geq n$ take *together* $O(\ell)$ time
3. Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

1. Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

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Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to **V**.
Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s, 0))
prev(s) ⇐ null
for each node u ≠ s do
    insert(Q, (u, ∞) )
    prev(u) ⇐ null

S = ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        if (dist(s, v) + ℓ(v, u) < dist(s, u) ) then
            decreaseKey(Q, (u, dist(s, v) + ℓ(v, u)) )
            prev(u) = v
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Shortest Path Tree

Dijkstra's algorithm finds the shortest path distances from s to $\mathbf{V}$.
Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s, 0))
prev(s) ⇐ null
for each node u ≠ s do
    insert(Q, (u, ∞) )
    prev(u) ⇐ null

S = ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        if (dist(s, v) + ℓ(v, u) < dist(s, u) ) then
            decreaseKey(Q, (u, dist(s, v) + ℓ(v, u)) )
            prev(u) = v
```

Shortest Path Tree

Lemma

The edge set $(\mathbf{u}, \mathrm{prev}(\mathbf{u}))$ *is the* reverse *of a shortest path tree rooted at* $\mathbf{s}$. *For each* $\mathbf{u}$, *the reverse of the path from* $\mathbf{u}$ *to* $\mathbf{s}$ *in the tree is a shortest path from* $\mathbf{s}$ *to* $\mathbf{u}$.

Proof Sketch.

1. The edge set $\{(\mathbf{u}, \mathrm{prev}(\mathbf{u})) \mid \mathbf{u} \in \mathbf{V}\}$ induces a directed in-tree rooted at $\mathbf{s}$ (Why?)
2. Use induction on $|\mathbf{S}|$ to argue that the tree is a shortest path tree for nodes in $\mathbf{V}$.

□

Shortest paths to s

Dijkstra's algorithm gives shortest paths from s to all nodes in V.
How do we find shortest paths from all of V to s?

1. In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
2. In directed graphs, use Dijkstra's algorithm in G^{rev}!

Shortest paths to $\mathbf{s}$

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