HW 10 (due Tuesday, at noon, December 2, 2014)

CS 473: Fundamental Algorithms, Fall 2014 Version: **1.01**

Make sure that you write the solutions for the problems on separate sheets of paper. Write your name and netid on each sheet.

Collaboration Policy: The homework can be worked in groups of up to 3 students each.

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1. (40 PTS.) Coolest path

After Kris the climber moved to Illinois, he started dating another climber girl, Oinoe, that he met at the climbing gym. Kris is a multi-dimensional personality and apart from climbing he also enjoys and excels at snowboarding. To his pleasant surprise, Oinoe was also a decent snowboarder and so they decided to take a trip to the backcountry mountains of New Zealand. They packed a few days worth of supplies, got their snowboarding gear on and hired a helicopter from the mountain base (hereafter referred to as the Base) to drop them off at the peak of a far away mountain called Death Peak. Kris and Oinoe's goal was to make their way from Death Peak back to the Base. The mountain was covered in powder and there was nobody else on it. The only thing that would reassure them that they were on a path back to the Base were some red poles planted in the snow at various parts of the mountain, called Stations. We can view the mountain as an undirected graph $G = (V, E)$ where each node is a Station and an edge (u, v) indicated that one can travel directly from station u to station v by snowboard (Kris and Oinoe carried with them a new kind of snowboard which was enhanced by a motor and allowed them to trave through flat parts of the mountain easily). The Death Peak is represented by a node s and the Base by a node t. Each edge e has a length $l_e \geq 0$ which represents distance from one Station to another. Also, some edges represent paths that are higher risk than others, in the sense that they are more avalanche-prone, have more tree-wells or obstacles along the way. So each edge e also has an integer risk $r_e \geq 0$, indicating the expected amount of damage in their health or equipment, if one traverses this edge.

It would be safest to travel by traversing the ridge of the mountain till they reach the end of the Sierra and then go downhill a very easy slope, but that would take them many days and they will run out of food. It would be fastest to just go down the steepest slope from Death Peak to the base of the mountain but that is very dangerous to create an avalanche. In general, for every path p from s to t, we define its total length to be the sum of the lengths of all its edges and its total risk to be the sum of the risks of all its edges.

Kris and Oinoe are looking for a complex type of shortest path in that graph that they name the *Coolest Path*: they need to get from s to t along a path whose total length and total risk is reasonably small. In concrete terms, the problem they want to solve is the following: given a graph with lengths and risks as above and integers L and R, is there a path from s to t whose total length is at most L and whose total risk is at most R?

Show that the Coolest Path problem is NP-Complete.

2. (20 PTS.) The world in base 3.

You are given an arithmetic formula F. This formula might use the constants $0, 1, 2$, additions, divisions, multiplications and subtractions (you can also use parenthesis). Naturally, the formula also contains free variables. To make things interesting, all the calculations in this formula are done modulo 3. As such, if $F \equiv x + y$, then for $x = 2$ and $y = 2$, the formula F

evaluates to $(2 + 2) \mod 3 = 1$.

In the Equation3 problem, given such formula, you have to decide whether or not there exists an assignment to the free variables of the formula such that it evaluates to 1.

Prove that this problem is NP-Complete by showing a polynomial time reduction from 3SAT to Equation3. (Don't forget also to show that Equation3 is in NP.)

3. (40 pts.) The strange strange world of MAX 2SAT.

(A) (2 pts.) Consider two boolean variables x and y. Write a 2CNF formula that computes the function $\neg(x \wedge y)$.

(B) (17 pts.) Given a connected graph $G = (V, E)$ with n vertices and m edges, we want to compute its maximum size independent set. To this end, we define a boolean variable for every vertex of V. Describe how to write a 2CNF formula that is true if and only if the vertices that are assigned value 1 are all independent.

(C) (10 pts.) Describe how to compute a 2CNF formula from G such that there is an assignment that satisfies at least Δ clauses of this formula if and only if there is an independent set in G of size k (or larger). To make things easy, you are allowed to duplicate the same clause in your formula as many times as you want. Naturally, the algorithm for computing this formula from G should work in polynomial time (and of course, you need to describe this algorithm). (The final formula size has to be polynomial in n and m.) What is the value of Δ as a function of n and m?

(D) (10 pts.) Using the above, prove that MAX 2SAT (i.e., given a 2CNF formula, compute the assignment that maximizes the number of clauses that are satisfied) is an NP-Hard problem. That is, show that if one can solve MAX 2SAT in polynomial time then one can solve 3SAT in polynomial time.

(E) (1 pts.) As you know, one can solve 2SAT in linear time. Why does this does not imply that MAX 2SAT can be solved in polynomial time?