HW 9 (due Tuesday, at noon, November 18, 2014)

CS 473: Fundamental Algorithms, Fall 2014 Version: **1.01**

Make sure that you write the solutions for the problems on separate sheets of paper. Write your name and netid on each sheet.

Collaboration Policy: The homework can be worked in groups of up to 3 students each.

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1. (50 PTS.) We want a proof!

The following question is long, but not very hard, and is intended to make sure you understand the following problems, and the basic concepts needed for proving NP-Completeness.

For each of the following problems, you are given an instance of the problem of size n. Imagine that the answer to this given instance is "yes". Imagine that you need to convince somebody that indeed the answer to the given instance is yes – to this end, describe:

(I) The format of the proof that the instance is correct.

(II) A bound on the length of the proof (its have to be of polynomial length in the input size).

(III) An efficient algorithm (as fast as possible [it has to be polynomial tie]) for verifying, given the instance and the proof, that indeed the given instance is indeed positive.

We solve the first such question, so that you understand what we want.[1]

(A) (0 PTS.)

Problem: Shortest Path

> *Instance:* A weighted undirected graph G, vertices s and t and a threshold w.
> *Question:* Is there a path between s and t in G of length at most w?

Solution: A "proof" in this case would be a path π in G (i.e., a sequence of at most n vertices) connecting s to t, such that its total weight is at most w. The algorithm to verify this solution, would verify that all the edges in the path are indeed in the graph, the path starts at s and ends at t, and that the total weight of the edges of the path is at most w. The proof has length $O(n)$ in this case, and the verification algorithm runs in $O(n^2)$ time. if we assume graph is given to us using an adjacency lists.

(B) (5 PTS.) **Problem:** Independent Set

> *Instance:* A graph G, integer k
> *Question:* Is there an independent set in G of size k?

(C) (5 PTS.) **Problem:** 3Colorable

> *Instance:* A graph G.
> *Question:* Is there a coloring of G using three colors?

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[1]We trust that the reader can by now readily translate all the following questions to questions about climbing vampires from Champaign. The reader can do this translation in their spare time for their own amusement.

(D) (5 PTS.) **Problem:** TSP

> *Instance:* A weighted undirected graph G, and a threshold w.
> *Question:* Is there a TSP tour of G of weight at most w?

(E) (5 PTS.) **Problem:** Vertex Cover

> *Instance:* A graph G, integer k
> *Question:* Is there a vertex cover in G of size k?

(F) (5 PTS.) **Problem:** Subset Sum

> *Instance:* S - set of positive integers,t: - an integer number (target).
> *Question:* Is there a subset $X \subseteq S$ such that $\sum_{x \in X} x = t$?

(G) (5 PTS.) **Problem:** 3DM

> *Instance:* X, Y, Z sets of n elements, and T a set of triples, such that $(a, b, c) \in T \subseteq X \times Y \times Z$.
> *Question:* Is there a subset $S \subseteq T$ of n disjoint triples, s.t. every element of $X \cup Y \cup Z$ is covered exactly once.?

(H) (5 PTS.) **Problem:** Partition

> *Instance:* A set S of n numbers.
> *Question:* Is there a subset $T \subseteq S$ s.t. $\sum_{t \in T} t = \sum_{s \in S \setminus T} s$?

(I) (5 PTS.) **Problem:** SET COVER

> *Instance:* $(X, \mathcal{F}, k)$:
> X: A set of n elements
> $\mathcal{F}$: A family of subsets of S, s.t. $\bigcup_{X \in \mathcal{F}} X = X$.
> k: A positive integer.
> *Question:* Are there k sets $S_1, \ldots, S_k \in \mathcal{F}$ that cover S. Formally, $\bigcup_i S_i = X$?

(J) (5 PTS.) **Problem:** CYCLE HATER.

> *Instance:* An undirected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, and an integer $k > 0$.
> *Question:* Is there a subset $X \subseteq \mathsf{V}$ of at most k vertices, such that all cycles in G contain at least one vertices of X.

(K) (5 PTS.) **Problem:** CYCLE LOVER.

> *Instance:* An undirected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, and an integer $k > 0$.
> *Question:* Is there a subset $X \subseteq \mathsf{V}$ of at most k vertices, such that all cycles in G contain at least two vertices of X.

2. (50 PTS.) Independence Matrix

Consider a $0-1$ matrix H with n_1 rows and n_2 columns. We refer to a row or a column of the matrix H as a line. We say that a set of 1's in the matrix H is *independent* if no two of them appear in the same line. We also say that a set of lines in the matrix is a *cover* of H if they include (i.e., "cover") all the 1's in the matrix. Using the max-flow min-cut theorem on an appropriately defined network, show that the maximum number of independent 1's equals the minimum number of lines in the cover.