HW 7 (due Tuesday, at noon, October 28, 2014)

CS 473: Fundamental Algorithms, Fall 2014 Version: **1.0**

Make sure that you write the solutions for the problems on separate sheets of paper. Write your name and netid on each sheet.

Collaboration Policy: The homework can be worked in groups of up to 3 students each.

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1. (40 PTS.) Surf's up.

After selecting its new president, the Inter-Galactic Surfing School (IGSS) decided to send a few teams to the surfing competition that happens every year in Kauai. It is decided that each team of surfers will satisfy the following: for every surfer in the school, either she has to be in the team or one of her friends has to be in the team. Such a team is called *valid*. The president of IGSS, trying to gain the esteem of the school memders, would like to send as many teams as possible to Kauai in order to maximize the chances of some IGSS team to win. Her goal is to find the maximum number (*surfout* number) of mutually disjoint valid teams that can compete.

Let $G = (V, E)$ be an undirected graph where V consists of all the surfers and an edge (u, v) indicates that surfer u is friends with surfer v. Let δ be the degree of a minimum degree node in G. It is easy to see that the surfout number of G is at most $(\delta + 1)$ since each valid team has to contain u or some neighbor of u where u is a node with degree δ. In this problem we will see that the surfout number of an association of n surfers where the representing graph G has minimum degree δ is at least as large as $\lceil \frac{\delta+1}{c \ln n} \rceil$ for some sufficient large universal constant c. Note that this guarantees to send only 1 valid team in the competition if $\delta < c \ln n$ (the entire group of surfers can be chosen as the team). Let $k = \lceil \frac{\delta+1}{c \ln n} \rceil$. Consider the following randomized algorithm. To each surfer u independently give a team shirt with number $g(u)$ written on it, that is chosen uniformly at random from the numbers $\{1, 2, \ldots, k\}$.

(A) (20 PTS.) For a fixed surfer v and a fixed number i show that with probability at least $1 - 1/n^2$ there is a surfer with shirt number i that is either v or a neighbor of v. Choose c sufficiently large to ensure this.

(B) (10 PTS.) Using the above show that for a fixed number i the set of surfers that are given shirts with number i form a valid surf team for G with probability at least $1 - 1/n$.

(C) (10 PTS.) Using the above two parts argue that the surfout number of G is at least k.

2. (30 PTS.) Random walk.

Consider a full binary tree of height h. You start from the root, and at every stage you flip a coin and go the left subtree with probability half (if you get a head), and to the right subtree with probability half (if you get a tail). You arrive to a leaf, and let's assume you took k turns to the left (and $h - k$ turns to the right) traversing from the root to this leaf. Then the value written in this leaf is α^k, where $\alpha < 1$ some parameter.

Let X_h be the random variable that is the returned value.

(A) (10 PTS.) Prove that $\mathrm{E}[X_h] = \left(\frac{1+\alpha}{2}\right)^h$ by stating a recursive formula on this value, and solving this recurrence. Alternatively, you can prove this by a direct calculation.

(B) (10 PTS.) Consider flipping a fair coin h times independently and interpret them as a path in the above tree. Let $\mathcal{E}$ be the event that we get at most $h/4$ heads in these coin flips. Argue that $\mathcal{E}$ happens if and only if $X_h \geq \alpha^{h/4}$.

(C) (10 PTS.) Markov's inequality states that for a positive random variable X we have that $P[X \geq t] \leq \mathrm{E}[X/t]$. Let Y be the number of heads when flipping a fair coin h times. Using Markov's inequality, (A) and (B) prove that

$$\Pr[\text{Out of } h \text{ coin flips getting at most } h/4 \text{ heads}] \leq \left(\frac{1+\alpha}{2\alpha^{1/4}}\right)^h.$$

In particular, by picking the appropriate value of α, prove that

$$\Pr[\text{Out of } h \text{ coin flips getting at most } h/4 \text{ heads}] \leq 0.88^h.$$

What is your value of α?

3. (30 PTS.) Conditional probabilities and expectations.

Assume there are two random variable X and Y, and you know the value of Y (say it is y). The *conditional probability* of X given Y, written as $\Pr[X \,|\, Y]$, is the probability of X getting the value x, given that you know that $Y = y$. Formally, it is

$$\Pr[X = x \,\big|\, Y = y] = \frac{\Pr[X = x \cap Y = y]}{\Pr[Y = y]}.$$

The *conditional expectation* of X given Y, written as $\mathrm{E}[X \,|\, Y = y]$ is the expected value of X if you know that $Y = y$. Formally, it is the function

$$f(y) = \mathrm{E}[X \,\big|\, Y = y] = \sum_{x \in \Omega} x \Pr[X = x \,\big|\, Y = y].$$

(A) (2 PTS.) Prove that if X and Y are independent then $\Pr[X = x \,|\, Y = y] = \Pr[X = x]$.

(B) (2 PTS.) Let X_i be the number of elements in **QuickSelect** in the ith recursive call, when starting with $X_0 = n$ elements. Prove that $\mathrm{E}[X_i \,|\, X_{i-1}] \leq (3/4)X_{i-1}$.

(C) (2 PTS.) Prove that for any discrete random variables X and Y it holds $\mathrm{E}[\mathrm{E}[X|Y]] = \mathrm{E}[X]$.

(D) (10 PTS.) Prove that, in expectation, the ith recursive call made by **QuickSelect** has at most $(3/4)^i n$ elements in the sub-array it is being called on.

(E) (4 PTS.) Let X be a random variable that can take on only non-negative values. Assume that $\mathrm{E}[X] = \mu$, where $\mu > 0$ is a real number (for example, μ might be 0.01). Prove that $\Pr[X \geq 1] \leq \mu$.

(F) (10 PTS.) Using (D) and (E) prove that with probability $\geq 1 - 1/n^{10}$ the depth of the recursion of **QuickSelect** when executed on an array with n elements is bounded by $M = c \lg n$, where c is some sufficiently larger constant (figure out the value of c for which your claim holds!).
(Hint: Consider the random variable which is the size of the subproblem that **QuickSelect** handles if it reaches the problem in depth M, and 0 if **QuickSelect** does not reach depth M in the recursion.)