CS 473: Fundamental Algorithms, Fall 2014
HW 1 (due Tuesday, at noon, September 9, 2014)

Version: 1.01.

This homework contains three problems. **Read the instructions for submitting homework on the course webpage**.

Collaboration Policy: For this homework, Problems 1–3 can be worked in groups of up to three students.

Note that the total number of points you can get for this homework is 110 out of 100, which means you are given a slack of 10 points :)

Each student individually have to also do **quiz 1** online.

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1. (30 PTS.) To surf or not to surf?

The inter-galactic surfing school (IGSS) has announced a new final exam policy for its students. The IGSS has n student surfers, and in order for them to pass the final exam, they need to surf one-by-one in front of the professors. The professors have allocated $m \geq n$ consecutive days when they are available to watch a student surf. Due to time constraints, there can only be one student surfing per day. The ith student s_i, is only free to surf two possible days, denoted by x_i and x'_i (since the rest of the days they are busy with other sports). Here, $x_i, x'_i \in \{1, \ldots, m\}$, for $i = 1, \ldots, n$. Given these constraints, design an algorithm that decides in polynomial time if there is a legal schedule for every surfing student to take the final, such that no two students surf the same day. To this end, answer the following.

(A) (5 PTS.) Consider the (directed) graph G with $2n$ nodes, where for every student there are two nodes $[i : x_i]$ or $[i : x'_i]$. For $\alpha \in \{x_i, x'_i\}$ and $\beta \in \{x_j, x'_j\}$, place an edge from $[i : \alpha]$ to $[j : \beta]$, if placing the ith student at day α implies that the jth student must be placed at day β because the other feasible placement of the jth student is at day α. How many edges can this graph have in the worst case? What is the running time of your algorithm to compute this graph?

(B) (5 PTS.) If there is a path in G from $[i : \alpha]$ to $[j : \beta]$, then we say that $s_i = \alpha$ ***forces*** $s_j = \beta$. Prove that if $s_i = x_i$ forces $s_j = x_j$ then the "reverse" must hold; that is, $s_j = x'_j$ forces $s_i = x'_i$.

(C) (5 PTS.) Prove that if for $x_i \neq x'_i$, $[i : x_i]$ and $[i : x'_i]$ are in the same strong connected component of G, then there is no legal way for all the students to take the final.

(D) (5 PTS.) Assume that there is a legal solution, and consider a strong connected component X of G involving students, say, $s_1, \ldots, s_t$ in G; that is, X is a set of vertices of the form $[1 : x_1], \ldots, [t : x_t]$. Then, prove that $[1 : x'_1], \ldots, [t : x'_t]$ form their own connected component in G. Let this component be the ***mirror*** of X.

(E) (5 PTS.) Prove that if X is a strong connected component of G that is a sink in the meta graph $\mathsf{G}^{\mathrm{SCC}}$, then the mirror of X is a source in the meta graph $\mathsf{G}^{\mathrm{SCC}}$.

(F) (5 PTS.) Consider the algorithm that takes the sink X of the meta-graph $\mathsf{G}^{\mathrm{SCC}}$, uses the associated slots as specified by the nodes in X, removes the vertices of X from G and the mirror of X from G, and repeats this process on the remaining graph. Prove that this algorithm generates a legal schedule for the students to take the final (or otherwise outputs that no such schedule exists). Also, describe how to implement this algorithm efficiently.

What is the running time of your algorithm in the worst case as a function of n and m.

2. (40 PTS.) Influences of vertices.

For a DAG G let $\mathrm{lp}(v, G)$ denote the longest directed path in G that starts in v (if there are several such paths, we arbitrarily choose one of them). Here, the length of a path is the number of edges in it. The ***influence*** of a vertex v in G, denoted by $i(v)$, is the number of edges of $\mathrm{lp}(v, G)$. Let G be a DAG. Two vertices x and y in G are ***unrelated*** if there is no directed path from x to y, and no directed path from y to x in G.

(A) (5 pts) Prove, that if the edge (x, y) is in G then $i(x) > i(y)$.

(B) (5 pts) Prove, that if there is a path between x and y in G then $i(x) > i(y)$.

(C) (5 pts) Conclude, that if there are k vertices in G that have all the same influence then they are all unrelated.

(D) (20 pts) Present a linear time algorithm that computes for each vertex in G its influence. Prove the correctness of your algorithm.

(E) (5 pts) Prove, that in a DAG either there is a path of length $\lfloor \sqrt{n} \rfloor$ or, alternatively, there are $\lfloor \sqrt{n} \rfloor$ vertices which are all unrelated to each other.
Present an algorithm that outputs this path, or the set of unrelated vertices. How fast is your algorithm?

3. (30 PTS.) Wishful graph.

Let $\mathsf{G} = (V, E)$ be a directed graph. Define a relation R on the nodes V as follows: uRv iff u can reach v or v can reach u.

(A) (5 PTS.) Is R an equivalence relation? If yes, give a proof, otherwise give an example to show it is false.

(B) (25 PTS.) Call G ***uselessly-connected*** if for every pair of nodes $u, v \in V$, we have that there is either a path from u to v in G, or a path from v to u in G. Give a linear time algorithm to determine if G is uselessly-connected, here linear time is $O(m + n)$, where $m = |E|$ and $n = |V|$.