CS 473: Fundamental Algorithms, Fall 2014
HW 0 (due at noon on Tuesday, September 2, 2014)

This homework contains two problems. **Read the instructions for submitting homework on the course webpage**.

You also have to do quiz 0 online!

Collaboration Policy: For this homework, each student should work independently and write up their own solutions and submit them.

Read the course policies before starting the homework.

- Homework 0 and Quiz 0 test your familiarity with prerequisite material: big-Oh notation, elementary algorithms and data structures, recurrences, graphs, and most importantly, induction, to help you identify gaps in your background knowledge. You are responsible for filling those gaps. The course web page has pointers to several excellent online resources for prerequisite material. If you need help, please ask in headbanging, on Piazza, in office hours, or by email.
- Each student must submit individual solutions for these homework problems. For all future homeworks, groups of up to three students may submit a single group solution for each problem.
- Please carefully read the course policies on the course web site. If you have any questions, please ask in lecture, in headbanging, on Piazza, in office hours, or by email. In particular:
  - Submit separately stapled solutions, one for each numbered problem, with your name and NetID clearly printed on each page.
  - You may use any source at your disposal: paper, electronic, human, or other, but you must write your solutions in your own words, and you must cite every source that you use (except for official course materials). Please see the academic integrity policy for more details.
  - No late homework will be accepted for any reason. However, we may forgive quizzes or homeworks in extenuating circumstances.
  - Answering "I don't know" to any (non-extra-credit) problem or subproblem, on any homework or exam, is worth 25% partial credit.
  - Algorithms or proofs containing phrases like and so on or repeat this process for all n, instead of an explicit loop, recursion, or induction, will receive a score of 0.
  - Unless explicitly stated otherwise, every homework problem requires a proof.

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1 Required problems

1. (50 PTS.) Lucas Numbers

The Lucas numbers L_n are defined recursively as follows:

$$L_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L_{n-2} + L_{n-1} & \text{otherwise.} \end{cases}$$

You may recognize this as the Fibonacci recurrence, but with a different base case ($L_0 = 2$ instead of $F_0 = 0$). Similarly, the anti-Lucas numbers Γ_n are defined recursively as follows:

$$\Gamma_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ \Gamma_{n-2} - \Gamma_{n-1} & \text{otherwise.} \end{cases}$$

The first few Luca numbers are as follows:

n	0	1	2	3	4	5	6	7
L_n	2	1	3	4	7	11	18	29

(a) Prove that $\Gamma_n = (-1)^{n-1} L_{n-1}$ for every positive integer n.

(b) Prove that any non-negative integer can be written as the sum of distinct non-consecutive Lucas numbers; that is, if L_i appears in the sum, then L_{i-1} and L_{i+1} cannot.

For example,

$$\begin{aligned} &4 = L_3 \\ &8 = 7 + 1 = L_4 + L_1 \\ &15 = 11 + 4 = L_5 + L_3 \\ &16 = 11 + 4 + 1 = L_5 + L_3 + L_1 \\ &23 = 18 + 4 + 1 = L_6 + L_3 + L_1 \end{aligned}$$

2. (50 PTS.) Tournament ranking.

A set of n tennis players $P_1, \ldots, P_n$ play in a tournament in which every player plays a match with every other player (a total of $\binom{n}{2}$ matches are played). There are no ties so for each pair of players (P_i, P_j), $i \neq j$ either P_i wins over P_j or vice-versa. We write $P_i \prec P_j$ if P_j wins against P_i in their match. We wish to rank the players from 1 to n with 1 being the best and n being the worst and justify the ranking. Let $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$ be a ranking of the players from 1 to n; here $i_1, i_2, \ldots, i_n$ is a permutation of $\{1, 2, \ldots, n\}$. A ranking is justified if $P_{i_n} \prec P_{i_{n-1}} \prec \ldots \prec P_{i_1}$. Prove the following via induction: for any integer n and *any* given outcomes of the $\binom{n}{2}$ matches, there is a ranking that is justified.