> *Ceterum in problematis natura fundatum est, ut methodi* quaecunque *continuo prolixiores evadant, quo maiores sunt numeri, ad quos applicantur; attamen pro methodis sequentibus difficultates perlente increscunt, numerique e septem, octos vel adeo adhuc pluribus figuris constantes praesertim per secundam felici semper successu tractati fuerunt, omnique celeritate, quam pro tantis numeris exspectare aequum est, qui secundum omnes methodos hactenus notas laborem, etiam calculatori indefatigabili intolerabilem, requirerent.*
>
> *[It is in the nature of the problem that* any *method will become more prolix as the numbers to which it is applied grow larger. Nevertheless, in the following methods the difficulties increase rather slowly, and numbers with seven, eight, or even more digits have been handled with success and speed beyond expectation, especially by the second method. The techniques that were previously known would require intolerable labor even for the most indefatigable calculator.]*
>
> — Carl Friedrich Gauß, *Disquisitiones Arithmeticae* (1801)
> English translation by A.A. Clarke (1965)

> *After much deliberation, the distinguished members of the international committee decided unanimously (when the Russian members went out for a caviar break) that since the Chinese emperor invented the method before anybody else had even been born, the method should be named after him. The Chinese emperor's name was Fast, so the method was called the Fast Fourier Transform.*
>
> — Thomas S. Huang, "How the fast Fourier transform got its name" (1971)

*2 Fast Fourier Transforms

2.1 Polynomials

In this lecture we'll talk about algorithms for manipulating *polynomials*: functions of one variable built from additions, subtractions, and multiplications (but no divisions). The most common representation for a polynomial $p(x)$ is as a sum of weighted powers of the variable x:

$$p(x) = \sum_{j=0}^{n} a_j x^j.$$

The numbers a_j are called the *coefficients* of the polynomial. The *degree* of the polynomial is the largest power of x whose coefficient is not equal to zero; in the example above, the degree is *at most* n. Any polynomial of degree n can be represented by an array $P[0..n]$ of $n+1$ coefficients, where $P[j]$ is the coefficient of the x^j term, and where $P[n] \neq 0$.

Here are three of the most common operations that are performed with polynomials:

- **Evaluate:** Give a polynomial p and a number x, compute the number $p(x)$.
- **Add:** Give two polynomials p and q, compute a polynomial $r = p + q$, so that $r(x) = p(x) + q(x)$ for all x. If p and q both have degree n, then their sum $p + q$ also has degree n.
- **Multiply:** Give two polynomials p and q, compute a polynomial $r = p \cdot q$, so that $r(x) = p(x) \cdot q(x)$ for all x. If p and q both have degree n, then their product $p \cdot q$ has degree $2n$.

We learned simple algorithms for all three of these operations in high-school algebra:

```
EVALUATE(P[0..n], x):
  X ← 1    ⟨⟨X = x^j⟩⟩
  y ← 0
  for j ← 0 to n
      y ← y + P[j] · X
      X ← X · x
  return y
```

```
ADD(P[0..n], Q[0..n]):
  for j ← 0 to n
      R[j] ← P[j] + Q[j]
  return R[0..n]
```

```
MULTIPLY(P[0..n], Q[0..m]):
  for j ← 0 to n + m
      R[j] ← 0
  for j ← 0 to n
      for k ← 0 to m
          R[j + k] ← R[j + k] + P[j] · Q[k]
  return R[0..n + m]
```

EVALUATE uses $O(n)$ arithmetic operations.[1] This is the best we can hope for, but we can cut the number of multiplications in half using *Horner's rule*:

$$p(x) = a_0 + x(a_1 + x(a_2 + \ldots + xa_n)).$$

```
HORNER(P[0..n], x):
  y ← P[n]
  for i ← n − 1 downto 0
      y ← x · y + P[i]
  return y
```

The addition algorithm also runs in $O(n)$ time, and this is clearly the best we can do.

The multiplication algorithm, however, runs in $O(n^2)$ time. In the previous lecture, we saw a divide and conquer algorithm (due to Karatsuba) for multiplying two n-bit integers in only $O(n^{\lg 3})$ steps; precisely the same algorithm can be applied here. Even cleverer divide-and-conquer strategies lead to multiplication algorithms whose running times are arbitrarily close to linear—$O(n^{1+\varepsilon})$ for your favorite value $e > 0$—but with great cleverness comes great confusion. These algorithms are difficult to understand, even more difficult to implement correctly, and not worth the trouble in practice thanks to large constant factors.

2.2 Alternate Representations

Part of what makes multiplication so much harder than the other two operations is our input representation. Coefficients vectors are the most common representation for polynomials, but there are at least two other useful representations.

2.2.1 Roots

The Fundamental Theorem of Algebra states that every polynomial p of degree n has exactly n *roots* $r_1, r_2, \ldots r_n$ such that $p(r_j) = 0$ for all j. Some of these roots may be irrational; some of these roots may by complex; and some of these roots may be repeated. Despite these complications, this theorem implies a unique representation of any polynomial of the form

$$p(x) = s \prod_{j=1}^{n} (x - r_j)$$

where the r_j's are the roots and s is a scale factor. Once again, to represent a polynomial of degree n, we need a list of $n + 1$ numbers: one scale factor and n roots.

Given a polynomial in this root representation, we can clearly evaluate it in $O(n)$ time. Given two polynomials in root representation, we can easily multiply them in $O(n)$ time by multiplying their scale factors and just concatenating the two root sequences.

Unfortunately, if we want to add two polynomials in root representation, we're out of luck. There's essentially *no* correlation between the roots of p, the roots of q, and the roots of $p + q$. We could convert the polynomials to the more familiar coefficient representation first—this takes $O(n^2)$ time using the high-school algorithms—but there's no easy way to convert the answer back. In fact, for most polynomials of degree 5 or more in coefficient form, it's *impossible* to compute roots exactly.[2]

[1] I'm going to assume in this lecture that each arithmetic operation takes $O(1)$ time. This may not be true in practice; in fact, one of the most powerful applications of fast Fourier transforms is fast *integer* multiplication. The fastest algorithm currently known for multiplying two n-bit integers, published by Martin Fürer in 2007, uses $O(n \log n \, 2^{O(\log^* n)})$ bit operations and is based on fast Fourier transforms.

[2] This is where numerical analysis comes from.

2.2.2 Samples

Our third representation for polynomials comes from a different consequence of the Fundamental Theorem of Algebra. Given a list of $n+1$ pairs $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$, there is *exactly one* polynomial p of degree n such that $p(x_j) = y_j$ for all j. This is just a generalization of the fact that any two points determine a unique line, because a line is the graph of a polynomial of degree 1. We say that the polynomial p ***interpolates*** the points (x_j, y_j). As long as we agree on the sample locations x_j in advance, we once again need exactly $n+1$ numbers to represent a polynomial of degree n.

Adding or multiplying two polynomials in this sample representation is easy, as long as they use the same sample locations x_j. To add the polynomials, just add their sample values. To multiply two polynomials, just multiply their sample values; however, if we're multiplying two polynomials of degree n, we must *start* with $2n+1$ sample values for each polynomial, because that's how many we need to uniquely represent their product. Both algorithms run in $O(n)$ time.

Unfortunately, evaluating a polynomial in this representation is no longer straightforward. The following formula, due to Lagrange, allows us to compute the value of any polynomial of degree n at any point, given a set of $n+1$ samples.

$$p(x) = \sum_{j=0}^{n-1} \left(\frac{y_j}{\prod_{k \neq j}(x_j - x_k)} \prod_{k \neq j} (x - x_k) \right)$$

Hopefully it's clear that formula actually describes a polynomial function of x, since each term in the sum is a scaled product of monomials. It's also not hard to verify that $p(x_j) = y_j$ for every index j; most of the terms of the sum vanish. As I mentioned earlier, the Fundamental Theorem of Algebra implies that p is *the only* polynomial that interpolates the points $\{(x_j, y_j)\}$. Lagrange's formula can be translated mechanically into an $O(n^2)$-time algorithm.

2.2.3 Summary

We find ourselves in the following frustrating situation. We have three representations for polynomials and three basic operations. Each representation allows us to almost trivially perform a different pair of operations in linear time, but the third takes at least quadratic time, if it can be done at all!

	evaluate	add	multiply
coefficients	$O(n)$	$O(n)$	$\boldsymbol{O(n^2)}$
roots + scale	$O(n)$	$\boldsymbol{\infty}$	$O(n)$
samples	$\boldsymbol{O(n^2)}$	$O(n)$	$O(n)$

2.3 Converting Between Representations

What we need are fast algorithms to convert quickly from one representation to another. That way, when we need to perform an operation that's hard for our default representation, we can switch to a different representation that makes the operation easy, perform that operation, and then switch back. This strategy immediately rules out the root representation, since (as I mentioned earlier) finding roots of polynomials is impossible in general, at least if we're interested in exact results.

So how do we convert from coefficients to samples and back? Clearly, once we choose our sample positions x_j, we can compute each sample value $y_j = p(x_j)$ in $O(n)$ time from the coefficients using Horner's rule. So we can convert a polynomial of degree n from coefficients to samples in $O(n^2)$ time.

Lagrange's formula can be used to convert the sample representation back to the more familiar coefficient form. If we use the naïve algorithms for adding and multiplying polynomials (in coefficient form), this conversion takes $O(n^3)$ time.

We can improve the cubic running time by observing that *both* conversion problems boil down to computing the product of a matrix and a vector. The explanation will be slightly simpler if we assume the polynomial has degree $n-1$, so that n is the number of coefficients or samples. Fix a sequence $x_0, x_1, \ldots, x_{n-1}$ of sample *positions*, and let V be the $n \times n$ matrix where $v_{ij} = x_i^j$ (indexing rows and columns from 0 to $n-1$):

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix}.$$

The matrix V is called a ***Vandermonde matrix***. The vector of coefficients $\vec{a} = (a_0, a_1, \ldots, a_{n-1})$ and the vector of sample *values* $\vec{y} = (y_0, y_1, \ldots, y_{n-1})$ are related by the matrix equation

$$V\vec{a} = \vec{y},$$

or in more detail,

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

Given this formulation, we can clearly transform any coefficient vector $\vec{a}$ into the corresponding sample vector $\vec{y}$ in $O(n^2)$ time.

Conversely, if we know the sample values $\vec{y}$, we can recover the coefficients by solving a system of n linear equations in n unknowns, which can be done in $O(n^3)$ time using Gaussian elimination.[3] But we can speed this up by implicitly hard-coding the sample positions into the algorithm, To convert from samples to coefficients, we can simply multiply the sample vector by the inverse of V, again in $O(n^2)$ time.

$$\vec{a} = V^{-1}\vec{y}$$

Computing V^{-1} would take $O(n^3)$ time if we had to do it from scratch using Gaussian elimination, but because we fixed the set of sample positions in advance, the matrix V^{-1} can be hard-coded directly into the algorithm.[4]

So we can convert from coefficients to samples and back in $O(n^2)$ time. At first lance, this result seems pointless; we can already add, multiply, or evaluate directly in either representation in $O(n^2)$ time, so why bother? But there's a degree of freedom we haven't exploited—***We get to choose the sample positions!*** Our conversion algorithm is slow only because we're trying to be too general. If we choose a set of sample positions with the right recursive structure, we can perform this conversion more quickly.

[3] In fact, Lagrange's formula is just a special case of Cramer's rule for solving linear systems.

[4] Actually, it is possible to invert an $n \times n$ matrix in $o(n^3)$ time, using fast matrix multiplication algorithms that closely resemble Karatsuba's sub-quadratic divide-and-conquer algorithm for integer/polynomial multiplication. On the other hand, my numerical-analysis colleagues have reasonable cause to shoot me in the face for daring to suggest, even in passing, that anyone actually invert a matrix at all, ever.

2.4 Divide and Conquer

Any polynomial of degree at most $n-1$ can be expressed as a combination of two polynomials of degree at most $(n/2)-1$ as follows:

$$p(x) = p_{\text{even}}(x^2) + x \cdot p_{\text{odd}}(x^2).$$

The coefficients of p_{even} are just the even-degree coefficients of p, and the coefficients of p_{odd} are just the odd-degree coefficients of p. Thus, we can evaluate $p(x)$ by recursively evaluating $p_{\text{even}}(x^2)$ and $p_{\text{odd}}(x^2)$ and performing $O(1)$ additional arithmetic operations.

Now call a set X of n values ***collapsing*** if either of the following conditions holds:

- X has one element.
- The set $X^2 = \{x^2 \mid x \in X\}$ has exactly $n/2$ elements and is (recursively) collapsing.

Clearly the size of any collapsing set is a power of 2. Given a polynomial p of degree $n-1$, and a collapsing set X of size n, we can compute the set $\{p(x) \mid x \in X\}$ of sample values as follows:

1. Recursively compute $\{p_{\text{even}}(x^2) \mid x \in X\} = \{p_{\text{even}}(y) \mid y \in X^2\}$.
2. Recursively compute $\{p_{\text{odd}}(x^2) \mid x \in X\} = \{p_{\text{odd}}(y) \mid y \in X^2\}$.
3. For each $x \in X$, compute $p(x) = p_{\text{even}}(x^2) + x \cdot p_{\text{odd}}(x^2)$.

The running time of this algorithm satisfies the familiar recurrence $T(n) = 2T(n/2) + \Theta(n)$, which as we all know solves to $T(n) = \Theta(n \log n)$.

Great! Now all we need is a sequence of arbitrarily large collapsing sets. The simplest method to construct such sets is just to invert the recursive definition: If X is a collapsible set of size n *that does not contain the number* 0, then $\sqrt{X} = \{\pm\sqrt{x} \mid x \in X\}$ is a collapsible set of size $2n$. This observation gives us an infinite sequence of collapsible sets, starting as follows:[5]

$$\begin{aligned}
X_1 &:= \{1\} \\
X_2 &:= \{1, -1\} \\
X_4 &:= \{1, -1, i, -i\} \\
X_8 &:= \left\{1, -1, i, -i, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right\}
\end{aligned}$$

2.5 The Discrete Fourier Transform

For any n, the elements of X_n are called the ***complex nth roots of unity***; these are the roots of the polynomial $x^n - 1 = 0$. These n complex values are spaced exactly evenly around the unit circle in the complex plane. Every nth root of unity is a power of the *primitive* nth root

$$\boxed{\omega_n = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.}$$

A typical nth root of unity has the form

$$\omega_n^k = e^{(2\pi i/n)k} = \cos\left(\frac{2\pi}{n}k\right) + i\sin\left(\frac{2\pi}{n}k\right).$$

These complex numbers have several useful properties for any integers n and k:

[5] In this lecture, i always represents the square root of -1. Computer scientists are used to thinking of i as an integer index into a sequence, an array, or a for-loop, but we obviously can't do that here. The physicist's habit of using $j = \sqrt{-1}$ just delays the problem (How do physicists write quaternions?), and typographical tricks like I or $\mathbf{i}$ or Mathematica's $\mathring{\mathrm{i}}$ are just stupid.

- There are exactly n different nth roots of unity: $\omega_n^k = \omega_n^{k \bmod n}$.
- If n is even, then $\omega_n^{k+n/2} = -\omega_n^k$; in particular, $\omega_n^{n/2} = -\omega_n^0 = -1$.
- $1/\omega_n^k = \omega_n^{-k} = \overline{\omega_n^k} = (\overline{\omega_n})^k$, where the bar represents complex conjugation: $\overline{a + bi} = a - bi$
- $\omega_n = \omega_{kn}^k$. Thus, every nth root of unity is also a (kn)th root of unity.

These properties imply immediately that if n is a power of 2, then the set of all nth roots of unity is collapsible!

If we sample a polynomial of degree $n-1$ at the nth roots of unity, the resulting list of sample values is called the ***discrete Fourier transform*** of the polynomial (or more formally, of its coefficient vector). Thus, given an array $P[0..n-1]$ of coefficients, its discrete Fourier transform is the vector $P^*[0..n-1]$ defined as follows:

$$P^*[j] := p(\omega_n^j) = \sum_{k=0}^{n-1} P[k] \cdot \omega_n^{jk}$$

As we already observed, the fact that sets of roots of unity are collapsible implies that we can compute the discrete Fourier transform in $O(n \log n)$ time. The resulting algorithm, called the ***fast Fourier transform***, was popularized by Cooley and Tukey in 1965.[6] The algorithm assumes that n is a power of two; if necessary, we can just pad the coefficient vector with zeros.

```
FFT(P[0..n-1]):
  if n = 1
    return P

  for j ← 0 to n/2 - 1
    U[j] ← P[2j]
    V[j] ← P[2j + 1]

  U* ← FFT(U[0..n/2 - 1])
  V* ← FFT(V[0..n/2 - 1])

  ω_n ← cos(2π/n) + i sin(2π/n)
  ω ← 1

  for j ← 0 to n/2 - 1
    P*[j]       ← U*[j] + ω · V*[j]
    P*[j + n/2] ← U*[j] - ω · V*[j]
    ω ← ω · ω_n

  return P*[0..n - 1]
```

Minor variants of this divide-and-conquer algorithm were previously described by Good in 1958, by Thomas in 1948, by Danielson and Lánczos in 1942, by Stumpf in 1937, by Yates in 1932, and by Runge in 1903; some special cases were published even earlier by Everett in 1860, by Smith in 1846, and by Carlini in 1828. But the algorithm, in its full modern recursive generality, was first *used* by Gauss around 1805 for calculating the periodic orbits of asteroids from a finite number of observations. In fact, Gauss's recursive algorithm predates even Fourier's introduction of harmonic analysis by two years. So, of course, the algorithm is universally called the *Cooley-Tukey* algorithm. Gauss's work built on earlier research on trigonometric interpolation by Bernoulli, Lagrange, Clairaut, and Euler; in particular, the

[6] Tukey apparently studied the algorithm to help detect Soviet nuclear tests without actually visiting Soviet nuclear facilities, by interpolating off-shore seismic readings. Without his rediscovery, the nuclear test ban treaty would never have been ratified, and we'd all be speaking Russian, or more likely, whatever language radioactive glass speaks.

first explicit description of the discrete "Fourier" transform was published by Clairaut in 1754, more than half a century before Fourier's work. Hooray for Stigler's Law![7]

2.6 Inverting the FFT

We also need to recover the coefficients of the product from the new sample values. Recall that the transformation from coefficients to sample values is *linear*; the sample vector is the product of a Vandermonde matrix V and the coefficient vector. For the discrete Fourier transform, each entry in V is an nth root of unity; specifically,

$$\boxed{v_{jk} = \omega_n^{jk}}$$

for all integers j and k. Thus,

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix}$$

To invert the discrete Fourier transform, converting sample values back to coefficients, we just have to multiply P^* by the inverse matrix V^{-1}. The following amazing fact implies that this is almost the same as multiplying by V itself:

Claim: $V^{-1} = \overline{V}/n$

Proof: We just have to show that $M = V\overline{V}$ is the identity matrix scaled by a factor of n. We can compute a single entry in M as follows:

$$m_{jk} = \sum_{l=0}^{n-1} \omega_n^{jl} \cdot \overline{\omega_n}^{lk} = \sum_{l=0}^{n-1} \omega_n^{jl-lk} = \sum_{l=0}^{n-1} (\omega_n^{j-k})^l$$

If $j = k$, then $\omega_n^{j-k} = \omega_n^0 = 1$, so

$$m_{jk} = \sum_{l=0}^{n-1} 1 = n,$$

and if $j \neq k$, we have a geometric series

$$m_{jk} = \sum_{l=0}^{n-1} (\omega_n^{j-k})^l = \frac{(\omega_n^{j-k})^n - 1}{\omega_n^{j-k} - 1} = \frac{(\omega_n^n)^{j-k} - 1}{\omega_n^{j-k} - 1} = \frac{1^{j-k} - 1}{\omega_n^{j-k} - 1} = 0. \qquad \square$$

[7] Lest anyone believe that Stigler's Law has treated Gauss unfairly, remember that "Gaussian elimination" was not discovered by Gauss; the algorithm was not even given that name until the mid-20th century! Elimination became the standard method for solving systems of linear equations in Europe in the early 1700s, when it appeared in an influential algebra textbook by Isaac Newton (published over his objections; he didn't want anyone to think it was his latest research). Although Newton apparently (and perhaps correctly) believed he had invented the elimination method, it actually appears in several earlier works, the oldest of which the eighth chapter of the Chinese manuscript *The Nine Chapters of the Mathematical Art*. The authors and precise age of the *Nine Chapters* are unknown, but commentary written by Liu Hui in 263CE claims that the text was already several centuries old. It was almost certainly *not* invented by a Chinese emperor named Fast.

In other words, if $W = V^{-1}$ then $w_{jk} = \overline{v_{jk}}/n = \overline{\omega_n^{jk}}/n = \omega_n^{-jk}/n$. What this means for us computer scientists is that any algorithm for computing the discrete Fourier transform can be easily modified to compute the inverse transform as well.

```
INVERSEFFT(P*[0..n-1]):
  if n = 1
    return P

  for j ← 0 to n/2 - 1
    U*[j] ← P*[2j]
    V*[j] ← P*[2j + 1]

  U ← INVERSEFFT(U[0..n/2 - 1])
  V ← INVERSEFFT(V[0..n/2 - 1])

  ω_n ← cos(2π/n) - i sin(2π/n)
  ω ← 1

  for j ← 0 to n/2 - 1
    P[j]       ← 2(U[j] + ω · V[j])
    P[j + n/2] ← 2(U[j] - ω · V[j])
    ω ← ω · ω_n

  return P[0..n - 1]
```

2.7 Fast Polynomial Multiplication

Finally, given two polynomials p and q, each represented by an array of coefficients, we can multiply them in $\Theta(n \log n)$ arithmetic operations as follows. First, pad the coefficient vectors and with zeros until the size is a power of two greater than or equal to the sum of the degrees. Then compute the DFTs of each coefficient vector, multiply the sample values one by one, and compute the inverse DFT of the resulting sample vector.

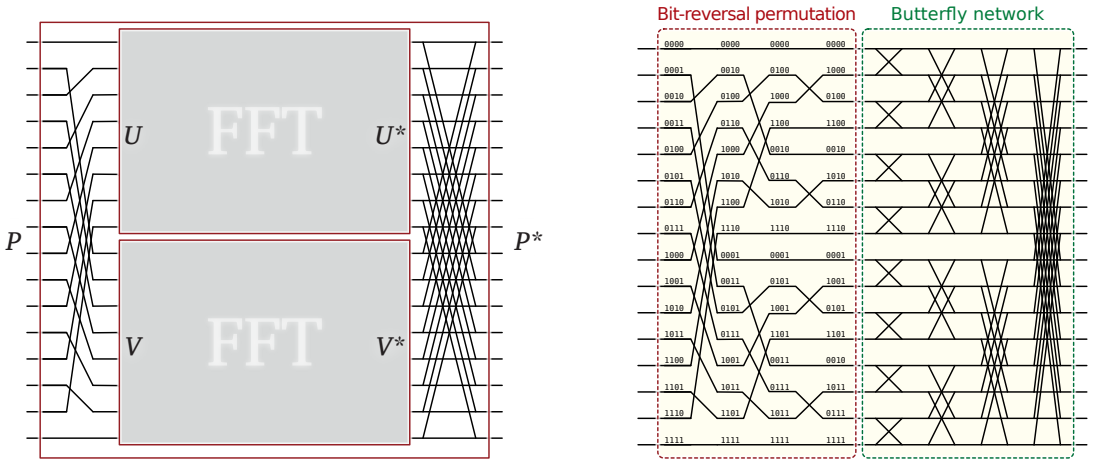
```
FFTMULTIPLY(P[0..n - 1], Q[0..m - 1]):
  ℓ ← ⌈lg(n + m)⌉
  for j ← n to 2^ℓ - 1
    P[j] ← 0
  for j ← m to 2^ℓ - 1
    Q[j] ← 0

  P* ← FFT(P)
  Q* ← FFT(Q)
  for j ← 0 to 2^ℓ - 1
    R*[j] ← P*[j] · Q*[j]
  return INVERSEFFT(R*)
```

2.8 Inside the FFT

FFTs are often implemented in hardware as circuits. To see the recursive structure of the circuit, let's connect the top-level inputs and outputs to the inputs and outputs of the recursive calls. On the left we split the input P into two recursive inputs U and V. On the right, we combine the outputs U^* and V^* to obtain the final output P^*.

If we expand this recursive structure completely, we see that the circuit splits naturally into two parts. The left half computes the *bit-reversal permutation* of the input. To find the position of $P[k]$ in this permutation, write k in binary, and then read the bits backward. For example, in an 8-element



The recursive structure of the FFT algorithm.

bit-reversal permutation, $P[3] = P[011_2]$ ends up in position $6 = 110_2$. The right half of the FFT circuit is a *butterfly network*. Butterfly networks are often used to route between processors in massively-parallel computers, because they allow any two processors to communicate in only $O(\log n)$ steps.

Exercises

1. For any two sets X and Y of integers, the Minkowski sum $X + Y$ is the set of all pairwise sums $\{x + y \mid x \in X, y \in Y\}$.

   (a) Describe an analyze and algorithm to compute the number of elements in $X+Y$ in $O(n^2 \log n)$ time. *[Hint: The answer is **not** always n^2.]*

   (b) Describe and analyze an algorithm to compute the number of elements in $X+Y$ in $O(M \log M)$ time, where M is the largest absolute value of any element of $X \cup Y$. *[Hint: What's this lecture about?]*

2. Suppose we are given a bit string $B[1..n]$. A triple of distinct indices $1 \le i < j < k \le n$ is called a ***well-spaced triple*** in B if $B[i] = B[j] = B[k] = 1$ and $k - j = j - i$.

   (a) Describe a brute-force algorithm to determine whether B has a well-spaced triple in $O(n^2)$ time.

   (b) Describe an algorithm to determine whether B has a well-spaced triple in $O(n \log n)$ time. *[Hint: Hint.]*

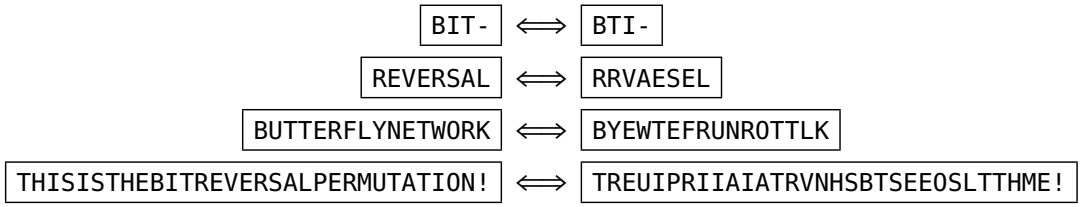
   (c) Describe an algorithm to determine the *number* of well-spaced triples in B in $O(n \log n)$ time.

3. (a) Describe an algorithm that determines whether a given set of n integers contains two elements whose sum is zero, in $O(n \log n)$ time.

   (b) Describe an algorithm that determines whether a given set of n integers contains *three* elements whose sum is zero, in $O(n^2)$ time.

   (c) Now suppose the input set X contains only integers between $-10000n$ and $10000n$. Describe an algorithm that determines whether X contains three elements whose sum is zero, in $O(n \log n)$ time. *[Hint: Hint.]*

4. Describe an algorithm that applies the bit-reversal permutation to an array $A[1..n]$ in $O(n)$ time when n is a power of 2.



5. The FFT algorithm we described in this lecture is limited to polynomials with 2^k coefficients for some integer k. Of course, we can always pad the coefficient vector with zeros to force it into this form, but this padding artificially inflates the input size, leading to a slower algorithm than necessary.

   Describe and analyze a similar DFT algorithm that works for polynomials with 3^k coefficients, by splitting the coefficient vector into three smaller vectors of length 3^{k-1}, recursively computing the DFT of each smaller vector, and correctly combining the results.