For a long time it puzzled me how something so expensive, so leading edge, could be so useless, and then it occurred to me that a computer is a stupid machine with the ability to do incredibly smart things, while computer programmers are smart people with the ability to do incredibly stupid things. They are, in short, a perfect match.

— Bill Bryson, *Notes from a Big Country* (1999)

23 Applications of Maximum Flow

23.1 Edge-Disjoint Paths

One of the easiest applications of maximum flows is computing the maximum number of edge-disjoint paths between two specified vertices s and t in a directed graph G using maximum flows. A set of paths in G is *edge-disjoint* if each edge in G appears in at most one of the paths; several edge-disjoint paths may pass through the same vertex, however.

If we give each edge capacity 1, then the maxflow from s to t assigns a flow of either 0 or 1 to every edge. Since any vertex of G lies on at most two saturated edges (one in and one out, or none at all), the subgraph S of saturated edges is the union of several edge-disjoint paths and cycles. Moreover, the number of paths is exactly equal to the value of the flow. Extracting the actual paths from S is easy—just follow any directed path in S from s to t, remove that path from S, and recurse.

Conversely, we can transform any collection of k edge-disjoint paths into a flow by pushing one unit of flow along each path from s to t; the value of the resulting flow is exactly k. It follows that any maxflow algorithm actually computes the largest possible set of edge-disjoint paths.

If we use Orlin's algorithm to compute the maximum (s,t)-flow, we can compute edge-disjoint paths in $\boldsymbol{O(VE)}$ ***time***, but Orlin's algorithm is overkill for this simple application. The cut $(\{s\}, V \setminus \{s\})$ has capacity at most $V-1$, so the maximum flow has value at most $V-1$. Thus, Ford and Fulkerson's original augmenting path algorithm also runs in $O(|f^*|E) = O(VE)$ time.

The same algorithm can also be used to find edge-disjoint paths in *undirected* graphs. We simply replace every undirected edge in G with a pair of directed edges, each with unit capacity, and compute a maximum flow from s to t in the resulting directed graph G' using the Ford-Fulkerson algorithm. For any edge uv in G, if our max flow saturates both directed edges $u{\to}v$ and $v{\to}u$ in G', we can remove *both* edges from the flow without changing its value. Thus, without loss of generality, the maximum flow assigns a direction to every saturated edge, and we can extract the edge-disjoint paths by searching the graph of directed saturated edges.

23.2 Vertex Capacities and Vertex-Disjoint Paths

Suppose we have capacities on the vertices as well as the edges. Here, in addition to our other constraints, we require that for any vertex v other than s and t, the total flow into v (and therefore the total flow out of v) is at most some non-negative value $c(v)$. How can we compute a maximum flow with these new constraints?

One possibility is to modify our existing algorithms to take these vertex capacities into account. Given a flow f, we can define the *residual capacity* of a vertex v to be its original capacity minus the total flow into v:

$$c_f(v) = c(v) - \sum_u f(u{\to}v).$$

Since we cannot send any more flow into a vertex with residual capacity 0 we remove from the residual graph G_f every edge $u \to v$ that appears in G whose head vertex v is saturated. Otherwise, the augmenting-path algorithm is unchanged.

But an even simpler method is to transform the input into a traditional flow network, with only edge capacities. Specifically, we replace every vertex v with two vertices v_{in} and v_{out}, connected by an edge $v_{\text{in}} \to v_{\text{out}}$ with capacity $c(v)$, and then replace every directed edge $u \to v$ with the edge $u_{\text{out}} \to v_{\text{in}}$ (keeping the same capacity). Finally, we compute the maximum flow from s_{out} to t_{in} in this modified flow network.

It is now easy to compute the maximum number of *vertex*-disjoint paths from s to t in any directed graph. Simply give every vertex capacity 1, and compute a maximum flow!

23.3 Maximum Matchings in Bipartite Graphs

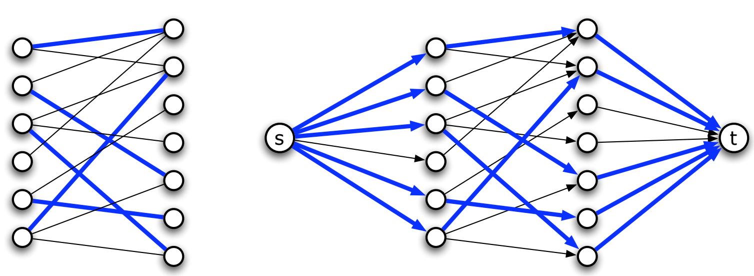
Another natural application of maximum flows is finding large *matchings* in bipartite graphs. A matching is a subgraph in which every vertex has degree at most one, or equivalently, a collection of edges such that no two share a vertex. The problem is to find the matching with the maximum number of edges in a given bipartite graph.

We can solve this problem by reducing it to a maximum flow problem as follows. Let G be the given bipartite graph with vertex set $U \cup W$, such that every edge joins a vertex in U to a vertex in W. We create a new *directed* graph G' by (1) orienting each edge from U to W, (2) adding two new vertices s and t, (3) adding edges from s to every vertex in U, and (4) adding edges from each vertex in W to t. Finally, we assign every edge in G' a capacity of 1.

Any matching M in G can be transformed into a flow f_M in G' as follows: For each edge uw in M, push one unit of flow along the path $s \to u \to w \to t$. These paths are disjoint except at s and t, so the resulting flow satisfies the capacity constraints. Moreover, the value of the resulting flow is equal to the number of edges in M.

Conversely, consider any (s,t)-flow f in G' computed using the Ford-Fulkerson augmenting path algorithm. Because the edge capacities are integers, the Ford-Fulkerson algorithm assigns an integer flow to every edge. (This is easy to verify by induction, hint, hint.) Moreover, since each edge has *unit* capacity, the computed flow either saturates ($f(e) = 1$) or avoids ($f(e) = 0$) every edge in G'. Finally, since at most one unit of flow can enter any vertex in U or leave any vertex in W, the saturated edges from U to W form a matching in G. The size of this matching is exactly $|f|$.

Thus, the size of the maximum matching in G is equal to the value of the maximum flow in G', and provided we compute the maxflow using augmenting paths, we can convert the actual maxflow into a maximum matching in $O(E)$ time. Again, we can compute the maximum flow in $\boldsymbol{O(VE)}$ ***time*** using either Orlin's algorithm or off-the-shelf Ford-Fulkerson.



A maximum matching in a bipartite graph G, and the corresponding maximum flow in G'.

23.4 Binary Assignment Problems

Maximum-cardinality matchings are a special case of a general family of so-called *assignment* problems.[1] An unweighted *binary* assignment problem involves two disjoint finite sets X and Y, which typically represent two different kinds of resources, such as web pages and servers, jobs and machines, rows and columns of a matrix, hospitals and interns, or customers and pints of ice cream. Our task is to choose the largest possible collection of pairs (x, y) as possible, where $x \in X$ and $y \in Y$, subject to several constraints of the following form:

- Each element $x \in X$ can appear in at most $c(x)$ pairs.
- Each element $y \in Y$ can appear in at most $c(y)$ pairs.
- Each pair $(x, y) \in X \times Y$ can appear in the output at most $c(x, y)$ times.

Each upper bound $c(x)$, $c(y)$, and $c(x, y)$ is either a (typically small) non-negative integer or ∞. Intuitively, we create each pair in our output by *assigning* an element of X to an element of y.

The maximum-matching problem is a special case, where $c(z) = 1$ for all $z \in X \cup Y$, and each $c(x, y)$ is either 0 or 1, depending on whether the pair xy defines an edge in the underlying bipartite graph.

Here is a slightly more interesting example. A nearby school, famous for its onerous administrative hurdles, decides to organize a dance. Every pair of students (one boy, one girl) who wants to dance must register in advance. School regulations limit each boy-girl pair to at most three dances together, and limits each student to at most ten dances overall. How can we maximize the number of dances? This is a binary assignment problem for the set X of girls and the set Y of boys, where for each girl x and boy y, we have $c(x) = c(y) = 10$ and either $c(x, y) = 3$ (if x and y registered to dance) or $c(x, y) = 0$ (if they didn't register).

Every binary assignment problem can be reduced to a standard maximum flow problem as follows. We construct a flow network $G = (V, E)$ with vertices $X \cup Y \cup \{s, t\}$ and the following edges:

- an edge $s \to x$ with capacity $c(x)$ for each $x \in X$,
- an edge $y \to t$ with capacity $c(y)$ for each $y \in Y$.
- an edge $x \to y$ with capacity $c(x, y)$ for each $x \in X$ and $y \in Y$, and

Because all the edges have integer capacities, the Ford-Fulkerson algorithm constructs an integer maximum flow f^*. This flow can be decomposed into the sum of $|f^*|$ paths of the form $s \to x \to y \to t$ for some $x \in X$ and $y \in Y$. For each such path, we report the pair (x, y). (Equivalently, the pair (x, y) appears in our output collection $f(x \to y)$ times.) It is easy to verify (hint, hint) that this collection of pairs satisfies all the necessary constraints. Conversely, any legal collection of r pairs can be transformed into a feasible integer flow with value r in G. Thus, the largest legal collection of pairs corresponds to a maximum flow in G. So our algorithm is correct.

Again, if we use Orlin's algorithm to compute the maximum flow, our assignment algorithm runs in $\boldsymbol{O(VE)}$ ***time***.

23.5 Baseball Elimination

Every year millions of baseball fans eagerly watch their favorite team, hoping they will win a spot in the playoffs, and ultimately the World Series. Sadly, most teams are "mathematically eliminated" days or even weeks before the regular season ends. Often, it is easy to spot when a team is eliminated—they

[1]Most authors refer to finding a maximum-weight matching in a bipartite graph as *the* assignment problem.

can't win enough games to catch up to the current leader in their division. But sometimes the situation is more subtle.

For example, here are the actual standings from the American League East on August 30, 1996.

Team	Won–Lost	Left	NYY	BAL	BOS	TOR	DET
New York Yankees	75–59	28		3	8	7	3
Baltimore Orioles	71–63	28	3		2	7	4
Boston Red Sox	69–66	27	8	2		0	0
Toronto Blue Jays	63–72	27	7	7	0		0
Detroit Lions	49–86	27	3	4	0	0	

Detroit is clearly behind, but some die-hard Lions fans may hold out hope that their team can still win. After all, if Detroit wins all 27 of their remaining games, they will end the season with 76 wins, more than any other team has now. So as long as every other team loses every game... but that's not possible, because some of those other teams still have to play each other. Here is one complete argument:[2]

> *By winning all of their remaining games, Detroit can finish the season with a record of 76 and 86. If the Yankees win just 2 more games, then they will finish the season with a 77 and 85 record which would put them ahead of Detroit. So, let's suppose the Tigers go undefeated for the rest of the season and the Yankees fail to win another game.*
>
> *The problem with this scenario is that New York still has 8 games left with Boston. If the Red Sox win all of these games, they will end the season with at least 77 wins putting them ahead of the Tigers. Thus, the only way for Detroit to even have a chance of finishing in first place, is for New York to win exactly one of the 8 games with Boston and lose all their other games. Meanwhile, the Sox must loss all the games they play agains teams other than New York. This puts them in a 3-way tie for first place....*
>
> *Now let's look at what happens to the Orioles and Blue Jays in our scenario. Baltimore has 2 games left with with Boston and 3 with New York. So, if everything happens as described above, the Orioles will finish with at least 76 wins. So, Detroit can catch Baltimore only if the Orioles lose all their games to teams other than New York and Boston. In particular, this means that Baltimore must lose all 7 of its remaining games with Toronto. The Blue Jays also have 7 games left with the Yankees and we have already seen that for Detroit to finish in first place, Toronto must will all of these games. But if that happens, the Blue Jays will win at least 14 more games giving them at final record of 77 and 85 or better which means they will finish ahead of the Tigers. So, no matter what happens from this point in the season on, Detroit can not finish in first place in the American League East.*

There has to be a better way to figure this out!

Here is a more abstract formulation of the problem. Our input consists of two arrays $W[1..n]$ and $G[1..n, 1..n]$, where $W[i]$ is the number of games team i has already won, and $G[i, j]$ is the number of upcoming games between teams i and j. We want to determine whether team n can end the season with the most wins (possibly tied with other teams).[3]

We model this question as an assignment problem: We want to ***assign*** *a winner to each game*, so that team n comes in first place. We have an assignment problem! Let $R[i] = \sum_j G[i, j]$ denote the number of remaining games for team i. We will assume that team n wins all $R[n]$ of its remaining games. Then team n can come in first place if and only if every other team i wins at most $W[n] + R[n] - W[i]$ of its $R[i]$ remaining games.

Since we want to ***assign*** winning teams to games, we start by building a bipartite graph, whose nodes represent the games and the teams. We have $\binom{n}{2}$ *game* nodes $g_{i,j}$, one for each pair $1 \le i < j < n$, and $n - 1$ *team* nodes t_i, one for each $1 \le i < n$. For each pair i, j, we add edges $g_{i,j} \to t_i$ and $g_{i,j} \to t_j$ with *infinite* capacity. We add a source vertex s and edges $s \to g_{i,j}$ with capacity $G[i, j]$ for each pair i, j. Finally, we add a target node t and edges $t_i \to t$ with capacity $W[n] - W[i] + R[n]$ for each team i.

[2]Both the example and this argument are taken from http://riot.ieor.berkeley.edu/~baseball/detroit.html.

[3]We assume here that no games end in a tie (always true for Major League Baseball), and that every game is actually played (not always true).

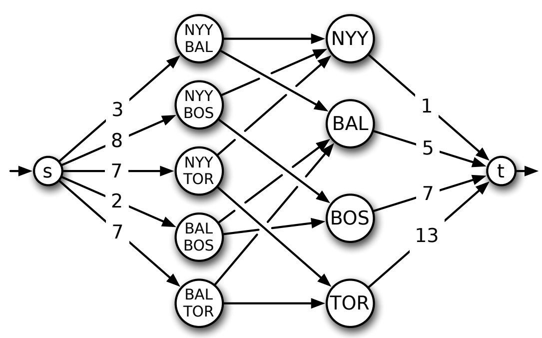
Theorem: *Team n can end the season in first place if and only if there is a feasible flow in this graph that saturates every edge leaving s.*

Proof: Suppose it is possible for team n to end the season in first place. Then every team $i < n$ wins at most $W[n]+R[n]-W[i]$ of the remaining games. For each game between team i and team j that team i wins, add one unit of flow along the path $s \to g_{i,j} \to t_i \to t$. Because there are exactly $G[i,j]$ games between teams i and j, every edge leaving s is saturated. Because each team i wins at most $W[n]+R[n]-W[i]$ games, the resulting flow is feasible.

Conversely, Let f be a feasible flow that saturates every edge out of s. Suppose team i wins exactly $f(g_{i,j} \to t_i)$ games against team j, for all i and j. Then teams i and j play $f(g_{i,j} \to t_i) + f(g_{i,j} \to t_j) = f(s \to g_{i,j}) = G[i,j]$ games, so every upcoming game is played. Moreover, each team i wins a total of $\sum_j f(g_{i,j} \to t_i) = f(t_i \to t) \leq W[n]+R[n]-W[i]$ upcoming games, and therefore at most $W[n]+R[n]$ games overall. Thus, if team n win all their upcoming games, they end the season in first place. $\square$

So, to decide whether our favorite team can win, we construct the flow network, compute a maximum flow, and report whether than maximum flow saturates the edges leaving s. The flow network has $O(n^2)$ vertices and $O(n^2)$ edges, and it can be constructed in $O(n^2)$ time. Using Orlin's algorithm, we can compute the maximum flow in $O(VE) = \boldsymbol{O(n^4)}$ ***time***.

The graph derived from the 1996 American League East standings is shown below. The total capacity of the edges leaving s is 27 (there are 27 remaining games), but the total capacity of the edges entering t is only 26. So the maximum flow has value at most 26, which means that Detroit is mathematically eliminated.

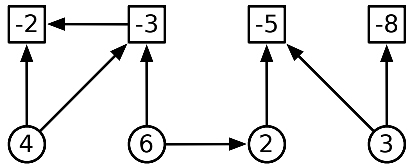


The flow graph for the 1996 American League East standings. Unlabeled edges have infinite capacity.

23.6 Project Selection

In our final example, suppose we are given a set of n projects that we could possibly perform; for simplicity, we identify each project by an integer between 1 and n. Some projects cannot be started until certain other projects are completed. This set of dependencies is described by a directed acyclic graph, where an edge $i \to j$ indicates that project i depends on project j. Finally, each project i has an associated *profit* p_i which is given to us if the project is completed; however, some projects have negative profits, which we interpret as positive *costs*. We can choose to finish any subset X of the projects that includes all its dependents; that is, for every project $x \in X$, every project that x depends on is also in X. Our goal is to find a valid subset of the projects whose total profit is as large as possible. In particular, if all of the jobs have negative profit, the correct answer is to do nothing.

At a high level, our task to partition the projects into two subsets S and T, the jobs we *Select* and the jobs we *Turn down*. So intuitively, we'd like to model our problem as a minimum cut problem in a certain



A dependency graph for a set of projects. Circles represent profitable projects; squares represent costly projects.

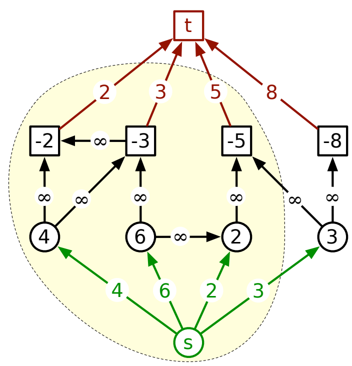
graph. But in which graph? How do we enforce prerequisites? We want to *maximize* profit, but we only know how to find *minimum* cuts. And how do we convert negative profits into positive capacities?

We define a new graph G by adding a source vertex s and a target vertex t to the dependency graph, with an edge $s \to j$ for every profitable job (with $p_j > 0$), and an edge $i \to t$ for every costly job (with $p_i < 0$). Intuitively, we can think of s as a new job ("To the bank!") with profit/cost 0 that we must perform last. We assign edge capacities as follows:

- $c(s \to j) = p_j$ for every profitable job j;
- $c(i \to t) = -p_i$ for every costly job i;
- $c(i \to j) = \infty$ for every dependency edge $i \to j$.

All edge-capacities are positive, so this is a legal input to the maximum cut problem.

Now consider an (s,t)-cut (S,T) in G. If the capacity $\|S,T\|$ is finite, then for every dependency edge $i \to j$, projects i and j are on the same side of the cut, which implies that S is a valid solution. Moreover, we claim that selecting the jobs in S earns us a total profit of $C - \|S,T\|$, where C is the sum of all the positive profits. This claim immediately implies that we can *maximize* our total profit by computing a *minimum* cut in G.



The flow network for the example dependency graph, along with its minimum cut.
The cut has capacity 13 and $C = 15$, so the total profit for the selected jobs is 2.

We prove our key claim as follows. For any subset A of projects, we define three functions:

$$\begin{aligned}
\mathit{cost}(A) &:= \sum_{i \in A:\, p_i < 0} -p_i = \sum_{i \in A} c(i \to t) \\
\mathit{benefit}(A) &:= \sum_{j \in A:\, p_i > 0} p_j = \sum_{j \in A} c(s \to j) \\
\mathit{profit}(A) &:= \sum_{i \in A} p_i = \mathit{benefit}(A) - \mathit{cost}(A).
\end{aligned}$$

By definition, $C = \mathit{benefit}(S) + \mathit{benefit}(T)$. Because the cut (S,T) has finite capacity, only edges of the form $s \to j$ and $i \to t$ can cross the cut. By construction, every edge $s \to j$ points to a profitable job and each edge $i \rightarrow t$ points from a costly job. Thus, $\|S, T\| = cost(S) + benefit(T)$. We immediately conclude that $C - \|S, T\| = benefit(S) - cost(S) = profit(S)$, as claimed.

Exercises

1. Given an undirected graph $G = (V, E)$, with three vertices u, v, and w, describe and analyze an algorithm to determine whether there is a path from u to w that passes through v.

2. Let $G = (V, E)$ be a directed graph where for each vertex v, the in-degree and out-degree of v are equal. Let u and v be two vertices G, and suppose G contains k edge-disjoint paths from u to v. Under these conditions, must G also contain k edge-disjoint paths from v to u? Give a proof or a counterexample with explanation.

3. Consider a directed graph $G = (V, E)$ with multiple source vertices $s_1, s_2, \ldots, s_\sigma$ and multiple target vertices $t_1, t_1, \ldots, t_\tau$, where no vertex is both a source and a target. A *multiterminal flow* is a function $f : E \to \mathbb{R}_{\geq 0}$ that satisfies the flow conservation constraint at every vertex that is neither a source nor a target. The value $|f|$ of a multiterminal flow is the total excess flow out of *all* the source vertices:
$$|f| := \sum_{i=1}^{\sigma} \left(\sum_w f(s_i \rightarrow w) - \sum_u f(u \rightarrow s_i) \right)$$
As usual, we are interested in finding flows with maximum value, subject to capacity constraints on the edges. (In particular, we don't care how much flow moves from any particular source to any particular target.)

   (a) Consider the following algorithm for computing multiterminal flows. The variables f and f' represent flow functions. The subroutine MaxFlow(G, s, t) solves the standard maximum flow problem with source s and target t.

```
MaxMultiFlow(G, s[1..σ], t[1..τ]):
  f ← 0                          ⟨⟨Initialize the flow⟩⟩
  for i ← 1 to σ
      for j ← 1 to τ
          f' ← MaxFlow(G_f, s[i], t[j])
          f ← f + f'             ⟨⟨Update the flow⟩⟩
  return f
```

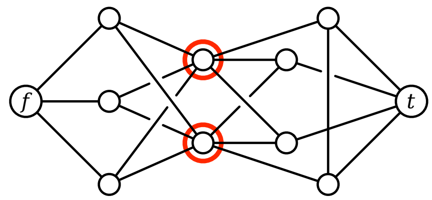
   Prove that this algorithm correctly computes a maximum multiterminal flow in G.

   (b) Describe a more efficient algorithm to compute a maximum multiterminal flow in G.

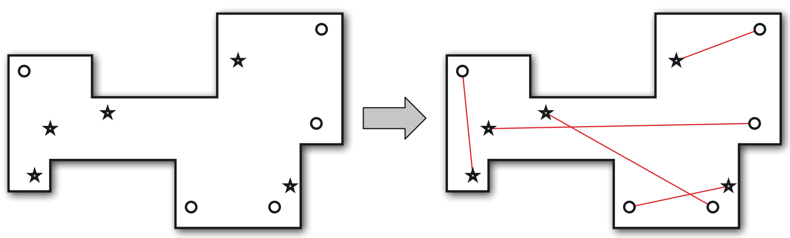
4. The Island of Sodor is home to a large number of towns and villages, connected by an extensive rail network. Recently, several cases of a deadly contagious disease (either swine flu or zombies; reports are unclear) have been reported in the village of Ffarquhar. The controller of the Sodor railway plans to close down certain railway stations to prevent the disease from spreading to Tidmouth, his home town. No trains can pass through a closed station. To minimize expense (and public notice), he wants to close down as few stations as possible. However, he cannot close the Ffarquhar station, because that would expose him to the disease, and he cannot close the Tidmouth station, because then he couldn't visit his favorite pub.

Describe and analyze an algorithm to find the minimum number of stations that must be closed to block all rail travel from Ffarquhar to Tidmouth. The Sodor rail network is represented by an undirected graph, with a vertex for each station and an edge for each rail connection between two stations. Two special vertices f and t represent the stations in Ffarquhar and Tidmouth.

For example, given the following input graph, your algorithm should return the number 2.



5. The UIUC Computer Science Department is installing a mini-golf course in the basement of the Siebel Center! The playing field is a closed polygon bounded by m horizontal and vertical line segments, meeting at right angles. The course has n *starting points* and n *holes*, in one-to-one correspondence. It is always possible hit the ball along a straight line directly from each starting point to the corresponding hole, without touching the boundary of the playing field. (Players are not allowed to bounce golf balls off the walls; too much glass.) The n starting points and n holes are all at distinct locations.



A minigolf course with five starting points ($\star$) and five holes ($\circ$), and a legal correspondence between them.

Sadly, the architect's computer crashed just as construction was about to begin. Thanks to the herculean efforts of their sysadmins, they were able to recover the *locations* of the starting points and the holes, but all information about which starting points correspond to which holes was lost!

Describe and analyze an algorithm to compute a one-to-one correspondence between the starting points and the holes that meets the straight-line requirement, or to report that no such correspondence exists. The input consists of the x- and y-coordinates of the m corners of the playing field, the n starting points, and the n holes. Assume you can determine in constant time whether two line segments intersect, given the x- and y-coordinates of their endpoints.

6. Suppose we are given a set of boxes, each specified by its height, width, and depth in centimeters. All three side lengths of each box lie strictly between 10cm and 20cm. As you should expect, one box can be placed inside another if the smaller box can be rotated so that its height, width, and depth are respectively smaller than the height, width, and depth of the larger box. Boxes can be nested recursively. Call a box is *visible* if it is not inside another box.

Describe and analyze an algorithm to nest the boxes so that the number of visible boxes is as small as possible.

7. The UIUC Faculty Senate has decided to convene a committee to determine whether Chief Illiniwek should become the official ~~mascot~~ *symbol* of the University of Illinois Global Campus.[4] Exactly one faculty member must be chosen from each academic department to serve on this committee. Some faculty members have appointments in multiple departments, but each committee member will represent only one department. For example, if Prof. Blagojevich is affiliated with both the Department of Corruption and the Department of Stupidity, and he is chosen as the Stupidity representative, then someone else must represent Corruption. Finally, University policy requires that any committee on virtual ~~mascots~~ *symbols* must contain the same number of assistant professors, associate professors, and full professors. Fortunately, the number of departments is a multiple of 3.

   Describe an efficient algorithm to select the membership of the Global Illiniwek Committee. Your input is a list of all UIUC faculty members, their ranks (assistant, associate, or full), and their departmental affiliation(s). There are n faculty members and $3k$ departments.

8. A *cycle cover* of a given directed graph $G = (V, E)$ is a set of vertex-disjoint cycles that cover all the vertices. Describe and analyze an efficient algorithm to find a cycle cover for a given graph, or correctly report that no cycle cover exists. *[Hint: Use bipartite matching!]*

9. You're organizing the First Annual UIUC Computer Science 72-Hour Dance Exchange, to be held all day Friday, Saturday, and Sunday. Several 30-minute sets of music will be played during the event, and a large number of DJs have applied to perform. You need to hire DJs according to the following constraints.

   - Exactly k sets of music must be played each day, and thus $3k$ sets altogether.
   - Each set must be played by a single DJ in a consistent music genre (ambient, bubblegum, dubstep, horrorcore, hyphy, trip-hop, Nitzhonot, Kwaito, J-pop, Nashville country, ...).
   - Each genre must be played at most once per day.
   - Each candidate DJ has given you a list of genres they are willing to play.
   - Each DJ can play at most three sets during the entire event.

   Suppose there are n candidate DJs and g different musical genres available. Describe and analyze an efficient algorithm that either assigns a DJ and a genre to each of the $3k$ sets, or correctly reports that no such assignment is possible.

10. The University of Southern North Dakota at Hoople has hired you to write an algorithm to schedule their final exams. Each semester, USNDH offers n different classes. There are r different rooms on campus and t different time slots in which exams can be offered. You are given two arrays $E[1..n]$ and $S[1..r]$, where $E[i]$ is the number of students enrolled in the ith class, and $S[j]$ is the number of seats in the jth room. At most one final exam can be held in each room during each time slot. Class i can hold its final exam in room j only if $E[i] < S[j]$.

    Describe and analyze an efficient algorithm to assign a room and a time slot to each class (or report correctly that no such assignment is possible).

[4]Thankfully, the Global Campus has faded into well-deserved obscurity, thanks in part to the 2009 admissions scandal. Imagine MOOCs, but with the same business model and faculty oversight as the University of Phoenix.

11. Suppose you are running a web site that is visited by the same set of people every day. Each visitor claims membership in one or more *demographic groups*; for example, a visitor might describe himself as male, 35–45 years old, a father, a resident of Illinois, an academic, a blogger, and a fan of Joss Whedon.[5] Your site is supported by advertisers. Each advertiser has told you which demographic groups should see its ads and how many of its ads you must show each day. Altogether, there are n visitors, k demographic groups, and m advertisers.

    Describe an efficient algorithm to determine, given all the data described in the previous paragraph, whether you can show each visitor exactly *one* ad per day, so that every advertiser has its desired number of ads displayed, and every ad is seen by someone in an appropriate demographic group.

12. Suppose we are given an array $A[1..m][1..n]$ of non-negative real numbers. We want to *round* A to an integer matrix, by replacing each entry x in A with either $\lfloor x \rfloor$ or $\lceil x \rceil$, without changing the sum of entries in any row or column of A. For example:

$$\begin{bmatrix} 1.2 & 3.4 & 2.4 \\ 3.9 & 4.0 & 2.1 \\ 7.9 & 1.6 & 0.5 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 4 & 2 \\ 4 & 4 & 2 \\ 8 & 1 & 1 \end{bmatrix}$$

    (a) Describe and analyze an efficient algorithm that either rounds A in this fashion, or reports correctly that no such rounding is possible.

    *(b) Suppose we are guaranteed that none of the entries in the input matrix A are integers. Describe and analyze an even faster algorithm that either rounds A or reports correctly that no such rounding is possible. For full credit, your algorithm must run in $O(mn)$ time. *[Hint: **Don't** use flows.]*

13. *Ad-hoc networks* are made up of low-powered wireless devices. In principle[6], these networks can be used on battlefields, in regions that have recently suffered from natural disasters, and in other hard-to-reach areas. The idea is that a large collection of cheap, simple devices could be distributed through the area of interest (for example, by dropping them from an airplane); the devices would then automatically configure themselves into a functioning wireless network.

    These devices can communicate only within a limited range. We assume all the devices are identical; there is a distance D such that two devices can communicate if and only if the distance between them is at most D.

    We would like our ad-hoc network to be reliable, but because the devices are cheap and low-powered, they frequently fail. If a device detects that it is likely to fail, it should transmit its information to some other *backup* device within its communication range. We require each device x to have k potential backup devices, all within distance D of x; we call these k devices the ***backup set*** of x. Also, we do not want any device to be in the backup set of too many other devices; otherwise, a single failure might affect a large fraction of the network.

    So suppose we are given the communication radius D, parameters b and k, and an array $d[1..n, 1..n]$ of distances, where $d[i, j]$ is the distance between device i and device j. Describe an algorithm that either computes a backup set of size k for each of the n devices, such that no

[5] Har har har! Mine is an evil laugh! Now *die!*
[6] but not really in practice

device appears in more than b backup sets, or reports (correctly) that no good collection of backup sets exists.

*14. A *rooted tree* is a directed acyclic graph, in which every vertex has exactly one incoming edge, except for the *root*, which has no incoming edges. Equivalently, a rooted tree consists of a root vertex, which has edges pointing to the roots of zero or more smaller rooted trees. Describe a polynomial-time algorithm to compute, given two rooted trees A and B, the largest common rooted subtree of A and B.

[Hint: Let $LCS(u, v)$ denote the largest common subtree whose root in A is u and whose root in B is v. Your algorithm should compute $LCS(u, v)$ for all vertices u and v using dynamic programming. This would be easy if every vertex had $O(1)$ children, and still straightforward if the children of each node were ordered from left to right and the common subtree had to respect that ordering. But for unordered trees with large degree, you need another trick to combine recursive subproblems efficiently. Don't waste your time trying to reduce the polynomial running time.]