Reductions and NP

Lecture 21

November 15, 2011

Part I

Reductions Continued

Polynomial Time Reduction

Karp reduction

A ***polynomial time reduction*** from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:

- given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
- $\mathcal{A}$ runs in time polynomial in $|I_X|$. This implies that $|I_Y|$ (size of I_Y) is polynomial in $|I_X|$
- Answer to I_X YES *iff* answer to I_Y is YES.

Notation: $X \leq_P Y$ if X reduces to Y

Proposition

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Such a reduction is called a ***Karp reduction***. Most reductions we will need are Karp reductions.

A More General Reduction

Turing Reduction

Definition (Turing reduction.)

Problem X polynomial time reduces to Y if there is an algorithm $\mathcal{A}$ for X that has the following properties:

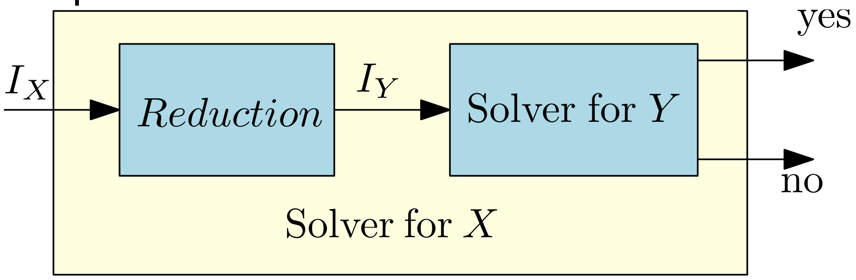
- on any given instance I_X of X, $\mathcal{A}$ uses polynomial in $|I_X|$ "steps"
- a step is either a standard computation step, or
- a sub-routine call to an algorithm that solves Y.

This is a ***Turing reduction***.

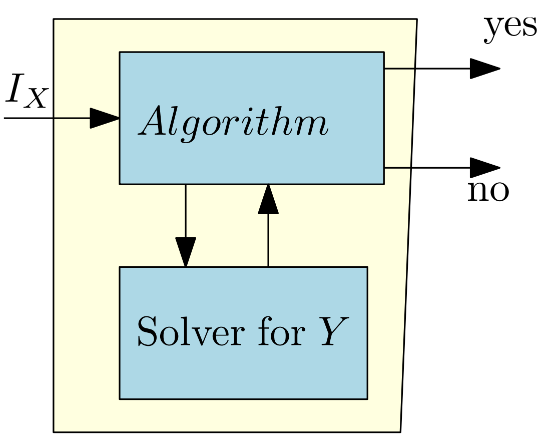
Note: In making sub-routine call to algorithm to solve Y, $\mathcal{A}$ can only ask questions of size polynomial in $|I_X|$. Why?

Comparing reductions

- Karp reduction:



- Turing reduction:



Turing reduction

- Algorithm to solve $\boldsymbol{X}$ can call solver for $\boldsymbol{Y}$ many times.
- Conceptually, every call to the solver of $\boldsymbol{Y}$ takes constant time.

Example of Turing Reduction

Input Collection of arcs on a circle.

Goal Compute the maximum number of non-overlapping arcs.

Reduced to the following problem:?

Input Collection of intervals on the line.

Goal Compute the maximum number of non-overlapping intervals.

How? Used algorithm for interval problem multiple times.

Turing vs Karp Reductions

- Turing reductions more general than Karp reductions.
- Turing reduction useful in obtaining algorithms via reductions.
- Karp reduction is simpler and easier to use to prove hardness of problems.
- Perhaps surprisingly, Karp reductions, although limited, suffice for most known NP-Completeness proofs.

Propositional Formulas

Definition

Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

- A ***literal*** is either a boolean variable x_i or its negation $\neg x_i$.
- A ***clause*** is a disjunction of literals.
  For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
- A ***formula in conjunctive normal form* (CNF)** is propositional formula which is a conjunction of clauses
  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.
- A formula φ is a **3CNF**:
  A CNF formula such that every clause has **exactly** 3 literals.
  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a 3CNF formula, but $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

Satisfiability

Problem: SAT

Instance: A CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Problem: 3SAT

Instance: A 3CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Satisfiability

SAT

Given a CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Example

$(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5$ is satisfiable; take $x_1, x_2, \ldots x_5$ to be all true

$(x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2)$ is not satisfiable

3SAT

Given a 3CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

(More on **2SAT** in a bit...)

Importance of SAT and 3SAT

- **SAT** and **3SAT** are basic constraint satisfaction problems.
- Many different problems can reduced to them because of the simple yet powerful expressively of logical constraints.
- Arise naturally in many applications involving hardware and software verification and correctness.
- As we will see, it is a fundamental problem in theory of NP-COMPLETEness.

SAT $\leq_P$ 3SAT

How SAT is different from 3SAT?

In **SAT** clauses might have arbitrary length: $1, 2, 3, \ldots$ variables:

$$\left(x \vee y \vee z \vee w \vee u\right) \wedge \left(\neg x \vee \neg y \vee \neg z \vee w \vee u\right) \wedge \left(\neg x\right)$$

In **3SAT** every clause must have ***exactly*** 3 different literals.

To reduce from an instance of **SAT** to an instance of **3SAT**, we must make all clauses to have exactly 3 variables...

Basic idea

- Pad short clauses so they have 3 literals.
- Break long clauses into shorter clauses.
- Repeat the above till we have a $3\mathrm{CNF}$.

3SAT $\leq_P$ SAT

- **3SAT** $\leq_P$ **SAT**.
- Because...
  A **3SAT** instance is also an instance of **SAT**.

SAT $\leq_P$ 3SAT

Claim

SAT $\leq_P$ ***3SAT***.

Given φ a **SAT** formula we create a **3SAT** formula φ' such that

- φ is satisfiable iff φ' is satisfiable
- φ' can be constructed from φ in time polynomial in $|\varphi|$.

Idea: if a clause of φ is not of length 3, replace it with several clauses of length exactly 3

SAT $\leq_P$ 3SAT

A clause with a single literal

Reduction Ideas

Challenge: Some of the clauses in φ may have less or more than 3 literals. For each clause with < 3 or > 3 literals, we will construct a set of logically equivalent clauses.

- Case clause with one literal: Let c be a clause with a single literal (i.e., $c = \ell$). Let u, v be new variables. Consider

$$c' = \left(\ell \vee u \vee v\right) \wedge \left(\ell \vee u \vee \neg v\right) \wedge \left(\ell \vee \neg u \vee v\right) \wedge \left(\ell \vee \neg u \vee \neg v\right).$$

Observe that c' is satisfiable iff c is satisfiable

SAT $\leq_P$ 3SAT

A clause with two literals

Reduction Ideas: 2 and more literals

- Case clause with 2 literals: Let $c = \ell_1 \vee \ell_2$. Let u be a new variable. Consider

$$c' = \left(\ell_1 \vee \ell_2 \vee u\right) \wedge \left(\ell_1 \vee \ell_2 \vee \neg u\right).$$

Again c is satisfiable iff c' is satisfiable

Breaking a clause

Lemma

For any boolean formulas X *and* Y *and* z *a new boolean variable. Then*

$$X \vee Y \text{ is satisfiable}$$

if and only if, z *can be assigned a value such that*

$$\Big(X \vee z\Big) \wedge \Big(Y \vee \neg z\Big) \text{ is satisfiable}$$

(with the same assignment to the variables appearing in X *and* Y*).*

$\mathbf{SAT} \leq_P \mathbf{3SAT}$ (contd)

Clauses with more than 3 literals

Let $c = \ell_1 \vee \cdots \vee \ell_k$. Let $u_1, \ldots u_{k-3}$ be new variables. Consider

$$\begin{aligned} c' = & \Big(\ell_1 \vee \ell_2 \vee u_1\Big) \wedge \Big(\ell_3 \vee \neg u_1 \vee u_2\Big) \\ & \wedge \Big(\ell_4 \vee \neg u_2 \vee u_3\Big) \wedge \\ & \cdots \wedge \Big(\ell_{k-2} \vee \neg u_{k-4} \vee u_{k-3}\Big) \wedge \Big(\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}\Big). \end{aligned}$$

Claim

c *is satisfiable iff* c' *is satisfiable.*

Another way to see it — reduce size of clause by one:

$$c' = \Big(\ell_1 \vee \ell_2 \ldots \vee \ell_{k-2} \vee u_{k-3}\Big) \wedge \Big(\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}\Big).$$

An Example

Example

$$\varphi = \Big(\neg x_1 \vee \neg x_4\Big) \wedge \Big(x_1 \vee \neg x_2 \vee \neg x_3\Big) \wedge \Big(\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1\Big) \wedge \Big(x_1\Big).$$

Equivalent form:

$$\begin{aligned}\psi = &(\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z)\\ &\wedge (x_1 \vee \neg x_2 \vee \neg x_3)\\ &\wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1)\\ &\wedge (x_1 \vee u \vee v) \wedge (x_1 \vee u \vee \neg v)\\ &\wedge (x_1 \vee \neg u \vee v) \wedge (x_1 \vee \neg u \vee \neg v).\end{aligned}$$

Overall Reduction Algorithm

Reduction from **SAT** to **3SAT**

```
ReduceSATTo3SAT(φ):
    // φ: CNF formula.
    for each clause c of φ do
        if c does not have exactly 3 literals then
            construct c' as before
        else
            c' = c
    ψ is conjunction of all c' constructed in loop
    return Solver3SAT(ψ)
```

Correctness (informal)

φ is satisfiable iff ψ is satisfiable because for each clause c, the new $3\mathrm{CNF}$ formula c' is logically equivalent to c.

What about 2SAT?

2SAT can be solved in polynomial time! (In fact, linear time!)

No known polynomial time reduction from **SAT** (or **3SAT**) to **2SAT**. If there was, then **SAT** and **3SAT** would be solvable in polynomial time.

Why the reduction from 3SAT to 2SAT fails?

Consider a clause $(x \lor y \lor z)$. We need to reduce it to a collection of $2\mathrm{CNF}$ clauses. Introduce a face variable α, and rewrite this as

$$
\begin{array}{lll}
 & (x \lor y \lor \alpha) \land (\neg\alpha \lor z) & \text{(bad! clause with 3 vars)} \\
\text{or} & (x \lor \alpha) \land (\neg\alpha \lor y \lor z) & \text{(bad! clause with 3 vars).}
\end{array}
$$

(In animal farm language: **2SAT** good, **3SAT** bad.)

What about 2SAT?

A challenging exercise: Given a **2SAT** formula show to compute its satisfying assignment...
(Hint: Create a graph with two vertices for each variable (for a variable x there would be two vertices with labels $x = 0$ and $x = 1$). For ever $2\mathrm{CNF}$ clause add two directed edges in the graph. The edges are implication edges: They state that if you decide to assign a certain value to a variable, then you must assign a certain value to some other variable.
Now compute the strong connected components in this graph, and continue from there...)

Independent Set

Problem: Independent Set

Instance: A graph G, integer k **Question:** Is there an independent set in G of size k?

3SAT $\leq_P$ Independent Set

The reduction 3SAT $\leq_P$ Independent Set

Input: Given a $3\mathrm{CNF}$ formula φ
Goal: Construct a graph G_φ and number k such that G_φ has an independent set of size k if and only if φ is satisfiable.
G_φ should be constructable in time polynomial in size of φ

Importance of reduction: Although **3SAT** is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

Notice: We handle only $3\mathrm{CNF}$ formulas – reduction would not work for other kinds of boolean formulas.

Interpreting 3SAT

There are two ways to think about **3SAT**

- Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
- Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of **3SAT** to construct the reduction.

The Reduction

- G_φ will have one vertex for each literal in a clause
- Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
- Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
- Take k to be the number of clauses

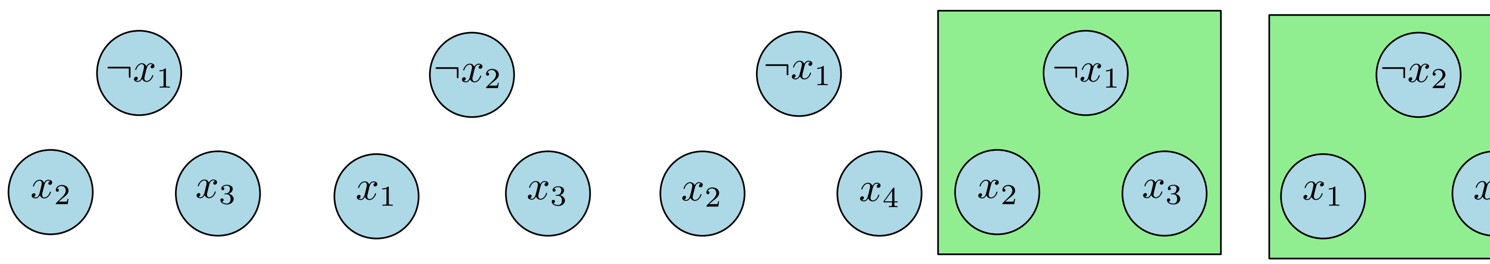


Figure: Graph for

Correctness

Proposition

φ is satisfiable iff G_φ has an independent set of size k ($=$ number of clauses in φ).

Proof.

$\Rightarrow$ Let a be the truth assignment satisfying φ

- Pick one of the vertices, corresponding to true literals under a, from each triangle. This is an independent set of the appropriate size $\square$

Correctness (contd)

Proposition

φ is satisfiable iff G_φ has an independent set of size k ($=$ number of clauses in φ).

Proof.

$\Leftarrow$ Let S be an independent set of size k

- S must contain exactly one vertex from each clause
- S cannot contain vertices labeled by conflicting clauses
- Thus, it is possible to obtain a truth assignment that makes in the literals in S true; such an assignment satisfies one literal in every clause $\square$

Transitivity of Reductions

Lemma

$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y
In other words show that an algorithm for Y implies an algorithm for X.

Part II

Definition of NP

Recap ...

Problems

- **Independent Set**
- **Vertex Cover**
- **Set Cover**
- **SAT**
- **3SAT**

Relationship

3SAT $\leq_P$ **Independent Set** $\overset{\leq_P}{\geq_P}$ **Vertex Cover** $\leq_P$ **Set Cover**
3SAT $\leq_P$ **SAT** $\leq_P$ **3SAT**

Problems and Algorithms: Formal Approach

Decision Problems

- Problem Instance: Binary string s, with size $|s|$
- Problem: A set X of strings on which the answer should be "yes"; we call these YES instances of X. Strings not in X are NO instances of X.

Definition

- A is an algorithm for problem X if $A(s) =$ "yes" iff $s \in X$
- A is said to have a polynomial running time if there is a polynomial $p(\cdot)$ such that for every string s, $A(s)$ terminates in at most $O(p(|s|))$ steps

Polynomial Time

Definition

Polynomial time (denoted P) is the class of all (decision) problems that have an algorithm that solves it in polynomial time

Example

Problems in P include

- Is there a shortest path from s to t of length $\leq k$ in G?
- Is there a flow of value $\geq k$ in network G?
- Is there an assignment to variables to satisfy given linear constraints?

Efficiency Hypothesis

A problem X has an efficient algorithm iff $X \in P$, that is X has a polynomial time algorithm.

Justifications:

- Robustness of definition to variations in machines.
- A sound theoretical definition.
- Most known polynomial time algorithms for "natural" problems have small polynomial running times.

Problems with no known polynomial time algorithms

Problems

- **Independent Set**
- **Vertex Cover**
- **Set Cover**
- **SAT**
- **3SAT**

There are of course undecidable problems (no algorithm at all!) but many problems that we want to solve are like above.

Question: What is common to above problems?

Efficient Checkability

Above problems share the following feature:

For any YES instance I_X of X there is a proof/certificate/solution that is of length poly($|I_X|$) such that given a proof one can efficiently check that I_X is indeed a YES instance

Examples:

- **SAT** formula φ: proof is a satisfying assignment
- **Independent Set** in graph G and k: a subset S of vertices

Certifiers

Definition

An algorithm $C(\cdot, \cdot)$ is a certifier for problem X if for every $s \in X$ there is some string t such that $C(s, t) =$ "yes", and conversely, if for some s and t, $C(s, t) =$ "yes" then $s \in X$.
The string t is called a certificate or proof for s

Efficient Certifier

C is an efficient certifier for problem X if there is a polynomial $p(\cdot)$ such that for every string s, $s \in X$ iff there is a string t with $|t| \leq p(|s|)$, $C(s, t) =$ "yes" and C runs in polynomial time

Example: Independent Set

- Problem: Does $G = (V, E)$ have an independent set of size $\geq k$?
  - Certificate: Set $S \subseteq V$
  - Certifier: Check $|S| \geq k$ and no pair of vertices in S is connected by an edge

Example: Vertex Cover

- Problem: Does G have a vertex cover of size $\leq k$?
  - Certificate: $S \subseteq V$
  - Certifier: Check $|S| \leq k$ and that for every edge at least one endpoint is in S

Example: SAT

- Problem: Does formula φ have a satisfying truth assignment?
  - Certificate: Assignment a of $0/1$ values to each variable
  - Certifier: Check each clause under a and say "yes" if all clauses are true

Example:Composites

- Problem: Is number s a composite?
  - Certificate: A factor $t \leq s$ such that $t \neq 1$ and $t \neq s$
  - Certifier: Check that t divides s (Euclid's algorithm)

Nondeterministic Polynomial Time

Definition

Nondeterministic Polynomial Time (denoted by NP) is the class of all problems that have efficient certifiers

Example

Independent Set, **Vertex Cover**, **Set Cover**, **SAT**, **3SAT**, Composites are all examples of problems in NP

Asymmetry in Definition of NP

Note that only YES instances have a short proof/certificate. NO instances need not have a short certificate.

Example: **SAT** formula φ. No easy way to prove that φ is NOT satisfiable!

More on this and co-NP later on.

P versus NP

Proposition

$P \subseteq NP$

For a problem in P no need for a certificate!

Proof.

Consider problem $X \in P$ with algorithm A. Need to demonstrate that X has an efficient certifier

- Certifier C on input s, t, runs $A(s)$ and returns the answer
- C runs in polynomial time
- If $s \in X$ then for every t, $C(s, t) =$ "yes"
- If $s \not\in X$ then for every t, $C(s, t) =$ "no" $\square$

Exponential Time

Definition

Exponential Time (denoted EXP) is the collection of all problems that have an algorithm which on input s runs in exponential time, i.e., $O(2^{\mathrm{poly}(|s|)})$

Example: $O(2^n)$, $O(2^{n \log n})$, $O(2^{n^3})$, ...

NP versus EXP

Proposition

$NP \subseteq EXP$

Proof.

Let $X \in NP$ with certifier C. Need to design an exponential time algorithm for X

- For every t, with $|t| \leq p(|s|)$ run $C(s,t)$; answer "yes" if any one of these calls returns "yes"
- The above algorithm correctly solves X (exercise)
- Algorithm runs in $O(q(|s| + |p(s)|)2^{p(|s|)})$, where q is the running time of C $\square$

Examples

- **SAT**: try all possible truth assignment to variables.
- **Independent Set**: try all possible subsets of vertices.
- **Vertex Cover**: try all possible subsets of vertices.

Is NP efficiently solvable?

We know $\boldsymbol{P} \subseteq \boldsymbol{NP} \subseteq \boldsymbol{EXP}$

Big Question

Is there are problem in $\boldsymbol{NP}$ that does not belong to $\boldsymbol{P}$? Is $\boldsymbol{P} = \boldsymbol{NP}$?

If P = NP . . .

Or: If pigs could fly then life would be sweet.

- Many important optimization problems can be solved efficiently.
- The RSA cryptosystem can be broken.
- No security on the web.
- No e-commerce . . .
- Creativity can be automated! Proofs for mathematical statement can be found by computers automatically (if short ones exist).

P versus NP

Status

Relationship between ***P*** and ***NP*** remains one of the most important open problems in mathematics/computer science.

Consensus: Most people feel/believe ***P*** $\neq$ ***NP***.

Resolving ***P*** versus ***NP*** is a Clay Millennium Prize Problem. You can win a million dollars in addition to a Turing award and major fame!