CS 473: Algorithms, Fall 2010
HBS 0

1. The following is an inductive proof of the statement that in every tree $T = (V(T), E(T))$, $|E(T)| = |V(T)| - 1$, i.e a tree with n vertices has $n - 1$ edges.

   Proof: The proof is by induction on $|V(T)|$.

   Base case: Base case is when $|V(T)| = 1$. A tree with a single vertex has no edge, so $|E(T)| = 0$. Therefore in this case the formula is true since $0 = 1 - 1$.

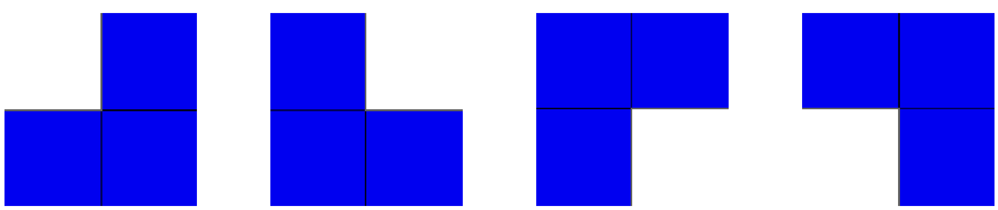
   Inductive step: Assume that the formula is true for all trees T where $|V(T)| = k$. We will prove that the formula is true for trees with $k + 1$ nodes. A tree T with $k + 1$ nodes can be obtained from a tree T' with k nodes by attaching a new vertex to a leaf of T'. This way we add exactly one vertex and one edge to T', so $|V(T)| = |V(T')| + 1$ and $|E(T)| = |E(T')| + 1$. Since $|V(T')| = k$ by induction hypothesis we have $|E(T')| = |V(T')| - 1$.

   Combining the last three relations we have $|E(T)| = |E(T')| + 1 = |V(T')| - 1 + 1 = |V(T)| - 1 - 1 + 1 = |V(T)| - 1$, which means that the formula is true for tree T.

   Show that the above is *not* a correct inductive proof! You must argue why it is not correct, and in particular produce a tree that the above argument does not cover.

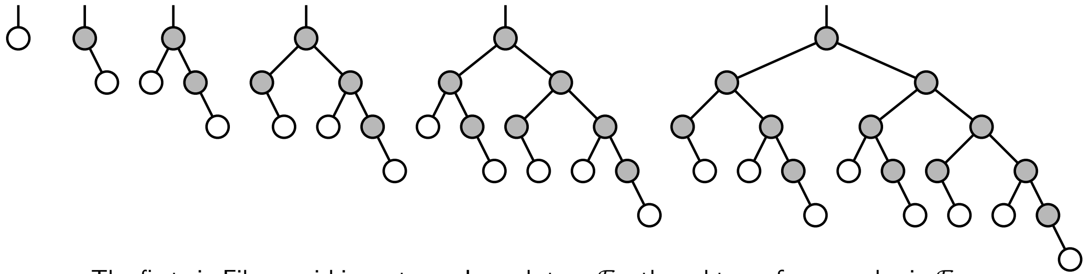
2. A k-coloring of a graph G is a labeling $f : V(G) \to S$ from vertices to colors where $|S| = k$. A k-coloring is proper if adjacent vertices are assigned different colors. A graph is k-colorable if it has a proper k-coloring. Prove that any graph G has a proper $(\Delta + 1)$-coloring where Δ is the maximum degree of a vertex of G (no vertex has more than Δ neighbors). For example, any cycle is 3-colorable as $\Delta = 2$ for cycles.

3. You are given a $2^n \times 2^n$ chessboard with a single square removed. Prove that you can tile the entire chessboard (minus the missing square) using copies of the 2×2 L's shown below.



4. The nth *Fibonacci binary tree* $\mathcal{F}_n$ is defined recursively as follows:

   - $\mathcal{F}_1$ is a single root node with no children.
   - For all $n \geq 2$, $\mathcal{F}_n$ is obtained from $\mathcal{F}_{n-1}$ by adding a right child to every leaf and adding a left child to every node that has only one child.



The first six Fibonacci binary trees. In each tree $\mathcal{F}_n$, the subtree of gray nodes is $\mathcal{F}_{n-1}$.

(a) Prove that the number of leaves in $\mathcal{F}_n$ is precisely the nth Fibonacci number: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

(b) How many nodes does $\mathcal{F}_n$ have?

(c) (*) What is the depth of $\mathcal{F}_n$'s most shallow leaf?