CS440/ECE448: Intro to Artificial Intelligence

# Lecture 9: More on predicate logic


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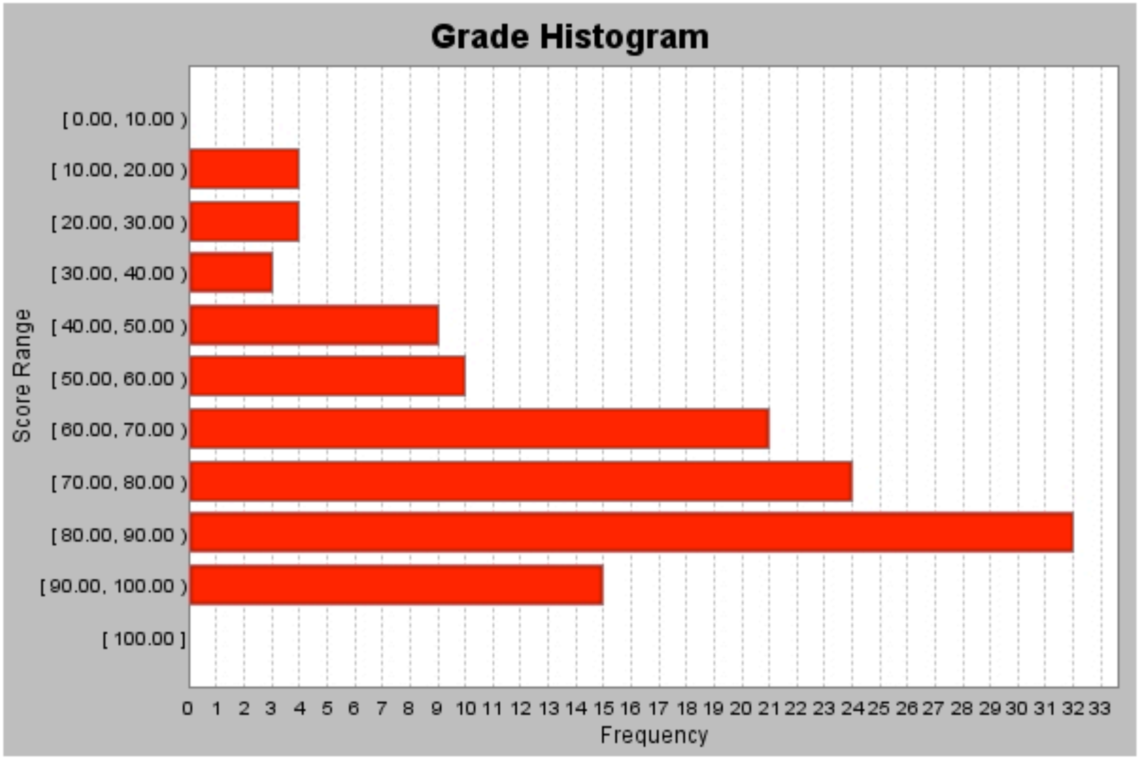
# Quick upgrade on quizzes

| | |
|---|---|
| Count: | **122** |
| Average: | **69.6** |
| Median: | **76.0** |
| Maximum: | **97.8** |
| Minimum: | **10.7** |
| Standard Deviation: | **20.37** |

Hide Histogram



# Review: syntax of predicate logic

# The building blocks

A (finite) set of **variables** $VAR$:

$$VAR = \{x, y, z, \ldots\}$$

A (finite) set of **constants** $CONST$:

$$CONST = \{john, mary, tom, \ldots\}$$

For $n=1\ldots N$ :

A (finite) set of $n$-place **function** symbols $FUNC$

$$FUNC_1 = \{fatherOf,\ \ successor, \ldots\}$$

A (finite) set of $n$-place **predicate** symbols $PRED_n$:

$$PRED_1 = \{student,\ \ blue, \ldots\}$$

$$PRED_2 = \{friend,\ \ sisterOf, \ldots\}$$

# Putting everything together

**Terms:** constants (*john*); variables (*x*); n-ary function symbols applied to *n* terms (*fatherOf(x)*)

- *Ground* terms contain no variables

**Formulas:** *n*-ary predicate symbols applied to *n* terms (*likes(x,y)*); negated formulas ($\neg$*fatherOf(x)*); conjunctions, disjunctions or implications of two formulas; quantified formulas

- *Ground* formulas (= sentences; propositions) contain no free variables
- *Open* formulas contain at least one free variable

# Semantics of predicate logic

# Model $M=(D,I)$

The **domain** $D$ is a nonempty set of objects:

$D = \{a1, b4, c8, \ldots\}$

The **interpretation function** $I$ maps:

- each constant $c$ to an element $c^I$ of $D$: $John^I = a1$
- each $n$-place function symbol $f$ to an (total) $n$-ary function $f^I\ D^n \rightarrow D$: $fatherOf^I(a1) = b4$
- each $n$-place predicate symbol $p$ to an $n$-ary relation $p^I \subseteq D^n$: $child^I = \{a1, c8\}$ $likes^I = \{\langle a1, b4\rangle, \langle b4, a1\rangle\}$

# Interpretation of variables

A variable assignment $v$ over a domain $D$ is a (partial) function from variables to $D$.

The assignment $v = [a21/x, b13/y]$ assigns object $a21$ to the variable $x$, and object $b13$ to variable $y$.

We recursively manipulate variable assignments when interpreting quantified formulas.
Notation: $v[b/z]$ is just like $v$, but it also maps $z$ to $b$.
We will make sure that $v$ is undefined for $z$.

# Interpretation of terms

**Variables:** $[\![x]\!]^{M,g} = g(x)$
defined by the variable assignment

**Constants:** $[\![c]\!]^{M,g} = c^I$
defined by the interpretation function

**Functions:** defined by the interpretation function and recursion on the arguments
$[\![f(t_1, \ldots. t_n)]\!]^{M,g} = f^{\mathrm{I}}([\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g})$

# Interpretation of formulas

## Atomic formulas:

$[\![P(t_1,\ldots t_n)]\!]^{M,g} = true$ iff $\langle [\![t_1]\!]^{M,g},\ldots [\![t_n]\!]^{M,g}\rangle \in P^I$

## Complex formulas (connectives):

$[\![\neg\, \varphi]\!]^{M,g} = true$ iff $[\![\varphi]\!]^{M,g} = false$

$[\![\varphi \wedge \psi]\!]^{M,g} = true$ iff $[\![\varphi]\!]^{M,g} = true$ and $[\![\psi]\!]^{M,g} = true$

$[\![\varphi \vee \psi]\!]^{M,g} = true$ iff $[\![\varphi]\!]^{M,g} = true$ or $[\![\psi]\!]^{M,g} = true$

$[\![\varphi \rightarrow \psi]\!]^{M,g} = true$ iff $[\![\varphi]\!]^{M,g} = false$ or $[\![\psi]\!]^{M,g} = true$

# Interpretation of formulas: quantifiers

**Universal** quantifier:

$[\![\forall x\ \varphi\ ]\!]^{M,g} = true$ iff $[\![\varphi]\!]^{M,g[u/x]} = true$

for **all** $u \in D$

**Existential** quantifier:

$[\![\forall x\ \varphi\ ]\!]^{M,g} = true$ iff $[\![\varphi]\!]^{M,g[u/x]} = true$

for at least **one** $u \in D$

# Satisfaction and entailment

$\varphi$ is satisfied in $M$ ($M \models \varphi$) *iff* $[\![\varphi]\!]^{M,[]} = true$

$\varphi$ is valid ($\models \varphi$) *iff* $\varphi$ is satisfied in any model.

$\varphi$ entails $\psi$ *iff* $\varphi \rightarrow \psi$ is valid.

NB: We cannot interpret *open* formulas
(i.e. containing free variables)

# Using predicate logic

# From English to predicate logic

Birds fly.

Some birds fly.

There are three people in SC1404.

SC1404 is empty.

There are exactly three people in SC1404.

# “Birds fly” vs “Some birds fly”

Birds fly:

$\forall x\ [\text{Bird}(x) \Rightarrow \text{Flies}(x)]$

$\equiv \neg \exists x\ [\text{Bird}(x) \wedge \neg \text{Flies}(x)]$

Some birds fly:

$\exists x\ [\text{Bird}(x) \wedge \text{Flies}(x)]$

# There are three people in SC1404
# SC1404 is empty.

There are three people in 1404:

$$\exists x \exists y \exists z\ [\text{person}(x) \wedge \text{in}(x, \text{SC1404})$$
$$\wedge\ \text{person}(y) \wedge \text{in}(x, \text{SC1404})$$
$$\wedge\ \text{person}(z) \wedge \text{in}(x, \text{SC1404})$$
$$\wedge\ x \neq y \wedge x \neq z \wedge y \neq z]$$

SC1404 is empty.

$$\neg \exists x\ [\text{person}(x) \wedge \text{in}(x, \text{SC1404})]$$

# There are ***exactly*** three people in SC1404

$$\exists x \exists y \exists z\ [\text{person}(x) \wedge \text{in}(x, SC1404)$$
$$\wedge\ \text{person}(y) \wedge \text{in}(x, SC1404)$$
$$\wedge\ \text{person}(z) \wedge \text{in}(x, SC1404)$$
$$\wedge\ x \neq y \wedge x \neq z \wedge y \neq z$$
$$\wedge\ \forall w\ [(\text{person}(w) \wedge \text{in}(w, SC1404)) \longrightarrow (w = x \vee w = y \vee w = z)]]$$

# What about….

*“Most birds fly.”*

This cannot be expressed in FOL:

$| \text{bird}^I \cap \text{fly}^I | > 0.5 \, | \text{bird}^I |$

# Inference in predicate logic

# Inference in predicate logic

*All men are mortal.*
*Socrates is a man.*
*Socrates is mortal.*

We need a new version of modus ponens:

$$\frac{\begin{array}{l}\forall x\ P(x) \longrightarrow Q(x) \\ \quad P(s')\end{array}}{Q(s')}$$

# FOL is semi-decidable

Can we prove whether A $\models$ B?

**Case 1:** A does entail B.
Any sound & complete inference procedure will terminate (and confirm that A $\models$ B)

**Case 2:** A does not entail B.
No inference procedure is guaranteed to terminate.

# How do we deal with quantifiers and variables?

Solution 1: **Propositionalization**
Ground all the variables.

Solution 2: **Lifted inference**
Ground (skolemize) all the existentially quantified variables. All remaining variables are universally quantified.
Use unification.

# Inference in predicate logic: propositionalization

# Grounding variables

$\theta = \{x_1/c_1, \ldots x_n/c_n\}$ is a list of substitutions: variable $x_i$ is replaced by ground term $c_{i.}$

The function $\textsc{Subst}(\theta, \psi)$ applies the substitutions in $\theta$ to the free variables in formula $\psi$ and returns the result.

$\textsc{Subst}(\{x/a, y/b\}, friend(x,y\})) = friend(a,b)$

# Substitutions

A *substitution* $\theta$ is a set of pairings of variables $v_i$ with terms $t_i$:

$$\theta = \{v_1/t_1, v_2/t_2, v_3/t_3, \ldots, v_n/ t_n\}$$

- Each variable $v_i$ is distinct
- $t_i$ can be any term (variable, constant, function), as long as it does not contain $v_i$ directly or indirectly

NB: the order of variables in $\theta$ doesn't matter

$\{x/y, y/f(a)\} = \{y/f(a), x/y\} = \{x/f(a), y/f(a)\}$

# Are these acceptable substitutions?

| | |
|---|---|
| $\{x/y\}$ | Yes |
| $\{x/y, z/F(y)\}$ | Yes |
| $\{x/y, z/F(y), x/A\}$ | NO: x appears twice |
| $\{x/y, z/F(y), y/A\}$ | Yes |
| $\{x/y, y/F(z), z/G(x)\}$ | NO: z/F(z) not allowed |

# Universal instantiation

We can replace a universally quantified variable with *any* ground term:

$$\frac{\forall x\ \psi(x)}{\textsc{Subst}(\{x/a\}, \psi)}\ (\mathrm{UI})$$

Hence: $$\frac{\forall x\ (\mathrm{man}(x) \longrightarrow \mathrm{mortal}(x))}{\textsc{Subst}(\{x/s'\}, \mathrm{man}(x) \longrightarrow \mathrm{mortal}(x))}\ (\mathrm{UI})$$

$$= \mathrm{man}(s') \longrightarrow \mathrm{mortal}(s')$$

# Existential instantiation

We can replace a existentially quantified variable with a *new* ground term:

$$\frac{\exists x\ \psi(x)}{\textsc{Subst}(\{x/t\}, \psi)}\ (\text{EI})$$

Condition:
$t$ doesn't appear in $\psi$ or in any other formula in our KB

Hence:

$$\frac{\exists x\ \text{man}(x)}{\text{man}(c')}\ (\text{EI})$$

# Propositionalization

If we have a finite domain, we can use UI and EI to eliminate all variables.

**KB:** $\forall x\ [(student(x) \wedge inClass(x)) \longrightarrow sleep(x)]$
$student(Mary)$ $professor(Julia)$

**Propositionalized KB:**
$(student(Mary) \wedge inClass(Mary)) \longrightarrow sleep(Mary)$
$(student(Julia) \wedge inClass(Julia)) \longrightarrow sleep(Julia)$
*etc.*

…this is not very efficient....

# Propositionalization

If we can reduce $A \models B$ to propositional resolution, $A \models B$ is decidable.

We cannot reduce $A \models B$ to propositional resolution if they contain functions or equality.

# Direct (lifted) inference in predicate logic: will it work?

# Some fragments of FOL are decidable

Can we prove whether A $\models$ B?

If both A and B belong to the same decidable fragment of FOL, any sound and complete inference procedure will terminate (and tell us whether A $\models$ B)

# Which fragment of FOL does $\varphi$ belong to?

Every WFF in FOL can be translated to prenex conjunctive normal form.

**Prenex form:**
All quantifiers are at the front of the formula

**Conjunctive Normal Form:**
A conjunction of clauses.

Depending on the form of the clauses in the CNF of $A$, $B$, $A \models B$ may be decidable.

# Prenex normal form

Every well-formed formula in FOL has an equivalent **prenex normal form,** in which all the quantifiers are at the front.

$\forall x \exists y \exists z \forall w\ \varphi(x,y,z,w)$

$\varphi(x,y,z,w)$ (the ‘matrix’) does not contain any quantifiers.

# We can move quantifiers outside of connectives

We already saw:

$$\forall x P(x) \land \forall y Q(y) \equiv \forall x \forall y\, [P(x) \land Q(y)]$$
$$\forall x P(x) \lor \forall y Q(y) \equiv \forall x \forall y\, [P(x) \lor Q(y)]$$

$$\exists x P(x) \land \exists y Q(y) \equiv \exists x \exists y\, [P(x) \land Q(y)]$$
$$\exists x P(x) \lor \exists y Q(y) \equiv \exists x \exists y\, [P(x) \lor Q(y)]$$

# Equivalences: alphabetic variants

We can move quantifiers to the front, e.g.:

$$\exists x P(x) \wedge \exists y Q(y) \equiv \exists x \exists y\ [P(x) \wedge Q(y)]$$

but we cannot 'merge' them:

$$\exists x P(x) \wedge \exists x Q(x) \not\equiv \exists x\ [P(x) \wedge Q(y)]$$

To avoid clashes, we first rename bound variables to any other new (unbound) variable:

$$\exists x P(x) \wedge \exists x Q(x) \equiv \exists x P(x) \wedge \exists y Q(y)$$

$\exists y Q(y)$ is an alphabetic variant of $\exists x Q(x)$

# We can move quantifiers outside of connectives

If $x$ is not free in $\varphi$:

$$\varphi \wedge \forall x\, \psi(x) \equiv \forall x[\varphi \wedge \psi(x)]$$
$$\varphi \wedge \exists x\, \psi(x) \equiv \exists x[\varphi \wedge \psi(x)]$$

$$\varphi \vee \forall x\, \psi(x) \equiv \forall x[\varphi \vee \psi(x)]$$
$$\varphi \vee \exists x\, \psi(x) \equiv \exists x[\varphi \vee \psi(x)]$$

$$\varphi \rightarrow \forall x\, \psi(x) \equiv \forall x[\varphi \rightarrow \psi(x)]$$
$$\varphi \rightarrow \exists x\, \psi(x) \equiv \exists x[\varphi \rightarrow \psi(x)]$$

# We can move quantifiers outside of negation

Not all x are $\psi$ = (At least) one x is not $\psi$

$$\neg [\forall x\ \psi(x)] \equiv \exists x [\neg \psi(x)]$$

No x is $\psi$ = All x are not $\psi$

$$\neg [\exists x\ \psi(x)] \equiv \forall x [\neg \psi(x)]$$

*Gold medal winners have not played against anybody who beat all other players.*

$\forall x[g(x) \rightarrow \neg \exists y[p(y,x) \wedge \forall z\, b(y,z)]]$

$\equiv \forall x[g(x) \rightarrow \forall y \neg [p(y,x) \wedge \forall z\, b(y,z)]]$
$\equiv \forall x[g(x) \rightarrow \forall y[\neg p(y,x) \vee \neg \forall z\, b(y,z)]]$
$\equiv \forall x[g(x) \rightarrow \forall y[\neg p(y,x) \vee \exists z \neg b(y,z)]]$
$\equiv \forall x \forall y\, [g(x) \rightarrow [\neg p(y,x) \vee \exists z \neg b(y,z)]]$
$\equiv \forall x \forall y\, [g(x) \rightarrow \exists z\, [\neg p(y,x) \vee \neg b(y,z)]]$

$\equiv \forall x \forall y \exists z\, [g(x) \rightarrow [\neg p(y,x) \vee \neg b(y,z)]]$
$\equiv \forall x \forall y \exists z\, [g(x) \rightarrow [p(y,x) \rightarrow \neg b(y,z)]]$

*= Everybody that a gold medal winner played against has not beaten (=has drawn or lost against) somebody.*

# Normal forms for FOL

**Conjunctive normal form:** a conjunction of clauses.

$(\varphi_1 \vee \ldots \vee \varphi_n) \wedge \ldots \wedge (\varphi'_1 \vee \ldots \vee \varphi'_m)$

**Clause:** a disjunction of an arbitrary number of positive or negative literals.

$(\varphi_1 \vee \neg\varphi_2 \vee \varphi_3 \vee \neg\varphi_4)$

**Literal**: a (negated) predicate and its arguments

likes(x, John')

¬likes(John', motherOf(Mary))

¬student(x)

# Fragments of FOL

**Definite clause:** exactly one positive literal
Facts: $\psi$
Implications: $\neg\varphi_1 \vee \ldots \vee \neg\varphi_n \vee \psi \equiv [(\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \psi]$

**Horn clause:** at most one positive literal.
All definite clauses, plus (resolution) queries: $\neg\psi$

**Clauses:** arbitrary number of positive literals
All Horn clauses, plus any disjunction, e.g $\varphi \vee \psi$

# Inference algorithms for FOL

**Forward chaining:**
Complete for definite clauses.

**Resolution:**
Refutation-complete for CNF.
(Resolution will derive a contradiction if a set of sentences is not satsifiable).

The resolution of two Horn clauses yields another Horn clause.

# Prerequisites for lifted inference: Skolemization and Unification

# Prerequisites for lifted inference

We cannot interpret open formulas, but we need to deal with quantified variables.

**Existentially quantified variables:**
replace by ground terms (Skolemization)

**Universally quantified variables:**
match with other universally quantified variables or ground terms (unification)

# Existentially quantified variables

If $\forall x\ \exists y\ R(x,y)$ is valid in our model, then:

For *any* way of instantiating $x$ there is a way to instantiate $y$ such that "$R$" holds between them.

Since $y$ can depend on $x$, we replace all occurrences of $y$ by a new function of $x$, $F(x)$. We assume $F(x)$ evaluates to the value which makes $R(x,F(x))$ true.
$\forall x\ \exists y\ R(x,y)$ is equivalent to $\forall x\ R(x,F(x))$

# **Skolemization:** remove existentially quantified variables

Replace any existentially quantified variable $\exists x$ that is in the scope of universally quantified variables $\forall y_1 \ldots \forall y_n$ with a new function $F(y_1,\ldots,y_n)$ (a **Skolem function**)

Replace any existentially quantified variable $\exists x$ that is not in the scope of any universally quantified variables with a new constant $c$ (a **Skolem term**)

# The effect of Skolemization

$\forall x\ \forall y\ \exists w\ \forall z\ Q(x, y, w, z, G(w, x))$

is equivalent to

$\forall x\ \forall y\ \forall z\ Q(x, y, P(x, y), z, G(P(x, y), x))$

where $P$ is the Skolem function for $w$.

NB: the Skolem function is a function,
so this is not decidable anymore.

# Modus ponens

With propositionalization:

$$\dfrac{\dfrac{\forall x\ \text{human}(x) \longrightarrow \text{mortal}(x)}{\text{human}(s') \longrightarrow \text{mortal}(s')}\ (\text{UI}) \qquad \text{human}(s')}{\text{mortal}(s')}\ (\text{MP})$$

How can we match $\text{human}(s')$ and $\forall x\ \text{human}(x) \longrightarrow \text{mortal}(x)$ directly?

# Substitutions

A *substitution* $\theta$ is a set of pairings of variables $v_i$ with terms $t_i$:

$$\theta = \{v_1/t_1, v_2/t_2, v_3/t_3, \ldots, v_n/ t_n\}$$

- Each variable $v_i$ is distinct
- $t_i$ can be any term (variable, constant, function), as long as it does not contain $v_i$ directly or indirectly

NB: the order of variables in $\theta$ doesn't matter

$\{x/y, y/f(a)\} = \{y/f(a), x/y\} = \{x/f(a), y/f(a)\}$

# Unification

Two sentences $\varphi$ and $\psi$ **unify** to $\sigma$

$(\text{UNIFY}(\varphi, \psi) = \sigma)$

if $\sigma$ is a substitution such that

$\text{SUBST}(\sigma,\varphi) = \text{SUBST}(\sigma,\psi)$.

Example:

UNIFY(like(x, M'), like(C',y)) ={x/C', y/M'}

# Unification

A set of sentences $\varphi_{1,}\ldots\varphi_n$ **unify** to $\sigma$
if for all $i \neq j$: $\textsc{Subst}(\sigma,\varphi_i) = \textsc{Subst}(\sigma, \varphi_j)$.

$\sigma$ is the unifier of $\varphi_{1,}\ldots\varphi_n$
$\textsc{Subst}(\sigma,\varphi_i)$ is a unification instance.

# Standardizing apart

Unification is not well-behaved if $\varphi$ and $\psi$ contain the same variable:

UNIFY(like(x, M'), like(C',x)): fail.

We need to *standardize* $\varphi$ and $\psi$ *apart* (rename this variable in one term):

UNIFY(like(x, M'), like(C',y)) = {x/C', y/M'}
to yield like(C',M')

# Do these unify?

**(Single lower case letters are variables)**

UNIFY(P(x,y,z), P(w, w, Fred))

$\sigma$ = {x=Fred, y=Fred, z=Fred, w=Fred}

Equivalently: $\sigma$' = {x=Fred, w=y, z=Fred , y=x}

Both yield P(Fred,Fred,Fred)

# Are there others?

UNIFY(P(x,y,z), P(w, w, Fred))

σ = {x=Mary, y=Mary, z=Fred, w=Mary}

Equivalently: σ' = {x=Mary, w=y, z=Fred , y=x}

Both yield   P(Mary, Mary, Fred)

# Most General Unifier (MGU)

$\sigma$ is the most general unifier (MGU) of $\varphi$ and $\psi$ if it imposes the fewest constraints.

The MGU of $\varphi$ and $\psi$ is unique.
(modulo alphabetic variants, i.e. different variable names)

Applying the MGU to an expression yields a *most general unification instance*.

We often define $\textsc{Unify}(\varphi, \psi)$ to return $\mathrm{MGU}(\varphi, \psi)$

# What is the MGU?

MGU(P(x,y,z), P(w,w,Fred))

$\sigma = \{x=w, y=w, z=Fred\}$ yields P(w,w,Fred)

Equivalently, $\sigma = \{x=u, y=u, w=u, z=Fred\}$
yields the alphabetic variant P(u,u,Fred)

# What is the MGU?

MGU( m(Ann, x, Bob), m(Ann, x, Bob) ):
m(Ann, x, Bob)
MGU( m(Ann, x, Bob), m(y, x, Chuck) ):
*fail*.
MGU( m(Ann, x, Bob), m(y, x, Father-of(Chuck) ):
*fail*.
MGU( p(w, w, Fred) , p(x, y, y) ):
p(Fred, Fred, Fred)
MGU( q(r, r), q(x, F(x)) ):
*fail*
MGU( r(g(x,Bob),y, y), r(z,g(Fred,w),z) ):
r(g(x,Bob), g(Fred,w), g(Fred(w)))

**To conclude…**

# Today's key concepts

**Semantics of first-order logic:**

Models for FOL

**Inference with first-order logic:**

Propositionalization
(Universal elimination and existential elimination)

**Dealing with variables:**

prenex normal form, unification and skolemization

***To do: read Ch.9.1-3***