CS 440: Introduction to AI

Homework 4 Solution

Due: November 9 11:59PM, 2010

Your answers must be concise and clear. Explain sufficiently that we can easily determine what you understand. We will give more points for a brief interesting discussion with no answer than for a bluffing answer.

Please email your solution to the TA at cs440ta@cs.illinois.edu.

1 Probability

1. [10 pts.] Consider a joint probability distribution $P(X, Y, Z)$ over variables X, Y, and Z. Assume that the probability of each outcome is non-zero. Is the following equation always true? Prove its correctness or incorrectness.

$$P(X \mid Y, Z) = \frac{P(Y \mid X, Z)P(X \mid Z)}{P(Y \mid Z)}$$

   (Solution) The given equation is always true. RHS of the equation has the numerator P(Y,X,Z)/P(Z) and has the denominator P(Y,Z)/P(Z) by Bayes rule. The resulting fraction P(Y,X,Z)/P(Y,Z) = P(X | Y,Z) by Bayes rule, and this is equivalent to the LHS of the equation.

2. [10 pts.] After your yearly checkup, the doctor has bad news and good news. The bad news is that you tested positive for a serious disease and that the test is 95% accurate (i.e., the probability of testing positive when you have the disease is 0.95 and the probability of testing negative when you don't have the disease is 0.95). The good news is that this is a rare disease, striking only 1 in 100,000 individuals. Why is it good news that the disease is rare? What are the chances that you actually have the disease?

   (Solution) The given probabilities are: $P(t \mid d) = 0.95$, $P(\neg t \mid \neg d) = 0.95$, and $P(d) = 0.00001$, where T and D are random variables for the test and the disease, respectively. Our goal is to calculate $P(d \mid t)$, the probability of actually having disease given the test is positive, and if the probability is quite low, then we can claim that it is good news.

$$
\begin{aligned}
P(d \mid t) &= \frac{P(t \mid d)P(d)}{P(t)} \\
&= \frac{P(t \mid d)P(d)}{P(t \mid d)P(d) + P(t \mid \neg d)P(\neg d)} \\
&= \frac{0.95 \cdot 0.00001}{0.95 \cdot 0.00001 + (1 - 0.95) \cdot 0.99999} \\
&\approx 0.00019
\end{aligned}
$$

2 Bayesian Networks

1. (a) [10/15 pts.] With the eight boolean variables, build a Bayesian Network so that the minmal number of parameters are needed to represent the conditional probabilities.

   (Solution) in Figure 1.

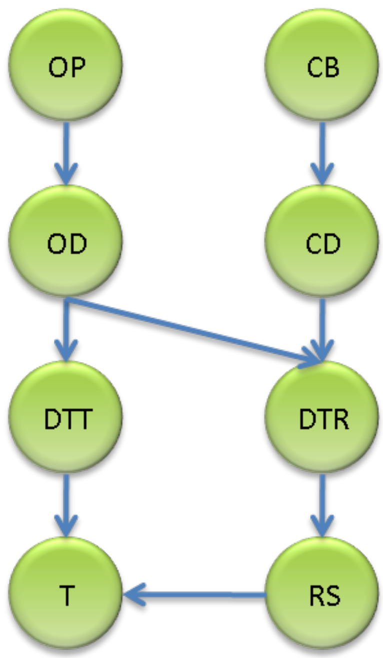
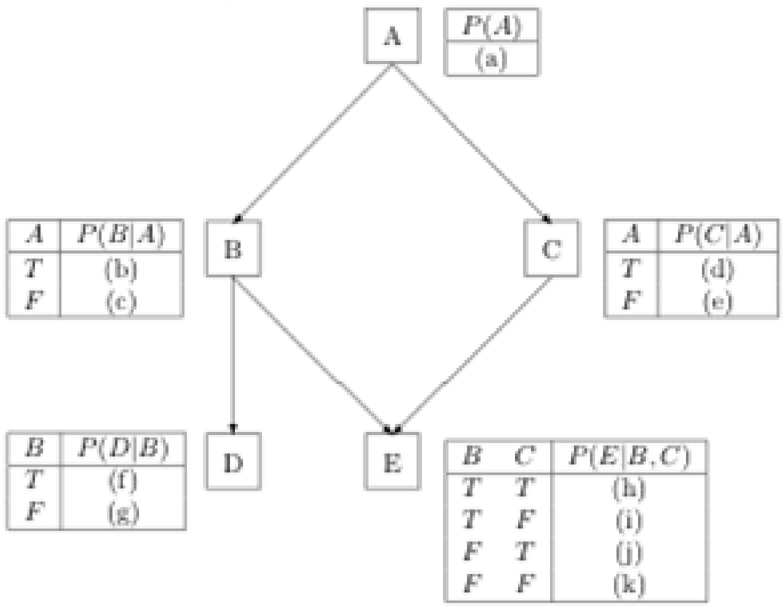


Figure 1: Solution

   (b) [5/15 pts.] How many parameters are needed to represent the conditional probabilities in (a)?

   (Solution) 1+1+2+2+2+4+4+2 = 18 parameters

2. [20 pts.] Given the following Bayesian network of Boolean variables:



and the joint probability table

		A				$\neg A$			
		B		$\neg B$		B		$\neg B$	
		C	$\neg C$	C	$\neg C$	C	$\neg C$	C	$\neg C$
D	E	0.0024	0.0504	0.0252	0.0196	0.01152	0.06912	0.03024	0.00672
	$\neg E$	0.0216	0.0056	0.0168	0.0784	0.10368	0.00768	0.02016	0.02688
$\neg D$	E	0.0036	0.0756	0.0108	0.0084	0.01728	0.10368	0.01296	0.00288
	$\neg E$	0.0324	0.0084	0.0072	0.0336	0.15552	0.01152	0.00864	0.01152

i. [4/20 pts.] Compute the probabilities (a) to (k) showing your work.
(a): 0.4 (b): 0.5 (c): 0.8 (d): 0.3 (e): 0.6 (f): 0.4 (g): 0.7 (h): 0.1 (i): 0.9 (j): 0.6 (k): 0.2

ii. [8/20 pts.] Calculate $P(D = T \mid C = T)$ (or $P(d \mid c)$) in terms of the probabilities (a) to (k). (Use the letters a to k and not the numbers that they represent).

$$\begin{aligned} P(d \mid c) &= \frac{P(d, c)}{P(c)} \\ &= \frac{\sum_{a,b,e} P(a, b, c, d, e)}{P(c)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sum_{a,b,e} P(a)P(b \mid a)P(c \mid a)P(d \mid b)P(e \mid b,c)}{P(c)} \\
&= \frac{\sum_{a} P(a)P(c \mid a)\sum_{b} P(b \mid a)P(d \mid b)\sum_{e} P(e \mid b,c)}{P(c)} \\
&= \frac{\sum_{a} P(a)P(c \mid a)\sum_{b} P(b \mid a)P(d \mid b) \cdot 1}{P(c \mid a)P(a) + P(c \mid \neg a)P(\neg a)}
\end{aligned}
$$

in terms of (a) - (k),

$$
\frac{a \cdot d \cdot (b \cdot f + (1-b) \cdot g) + (1-a) \cdot e \cdot (c \cdot f + (1-c) \cdot g)}{d \cdot a + e \cdot (1-a)}
$$

iii. [2/20 pts.] Use the results of (i) and (ii) to obtain a value for the probability.

$P(d \mid c) = P(d,c)/P(c)$

$P(d,c) = 0.4 \cdot 0.3 \cdot (0.5 \cdot 0.4 + 0.5 \cdot 0.7) + 0.6 \cdot 0.6 \cdot (0.8 \cdot 0.4 + 0.2 \cdot 0.7)$

$P(c) = 0.3 \cdot 0.4 + 0.6 \cdot 0.6$

$P(d \mid c) = 0.2316/0.48 = 0.4825$

iv. [2/20 pts.] Use the values from the joint probability table to check that (ii) and (iii) are correct.

$P(d \mid c) = P(d,c)/P(c)$

$P(d,c) = 0.0024 + 0.0216 + 0.0252 + 0.0168 + 0.01152 + 0.10368 + 0.03024 + 0.02016 = 0.2316$

$P(c) = 0.0024 + 0.0216 + 0.0036 + 0.0324 + 0.0252 + 0.0168 + 0.0108 + 0.0072 + 0.01152 + 0.10368 + 0.01728 + 0.15552 + 0.03024 + 0.02016 + 0.01296 + 0.00864 = 0.48$

$P(d \mid c) = 0.2316/0.48 = 0.4825$

v. [4/20 pts.] As you did in (ii), calculate $P(A = F, B = T \mid D = T)$ (or $P(\neg a, b \mid d)$) in terms of the probabilities (a) to (k). (Use the letters a to k and not the number that they represent).

$$
\begin{aligned}
P(\neg a, b \mid d) &= \frac{P(\neg a, b, d)}{P(d)} \\
&= \frac{\sum_{c,e} P(\neg a, b, c, d, e)}{P(d)} \\
&= \frac{\sum_{c,e} P(\neg a)P(b \mid \neg a)P(c \mid \neg a)P(d \mid b)P(e \mid b,c)}{P(d)} \\
&= \frac{P(\neg a)P(b \mid \neg a)P(d \mid b)\sum_{c} P(c \mid \neg a)\sum_{e} P(e \mid b,c)}{P(d)} \\
&= \frac{P(\neg a)P(b \mid \neg a)P(d \mid b) \cdot 1 \cdot 1}{P(d)}
\end{aligned}
$$

where $P(d)$ is

$$\begin{aligned} P(d) &= \sum_{a,b} P(a,b,d) \\ &= \sum_{a,b} P(a)P(b \mid a)P(d \mid b) \\ &= \sum_{b} P(d \mid b) \sum_{a} P(a)P(b \mid a) \end{aligned}$$

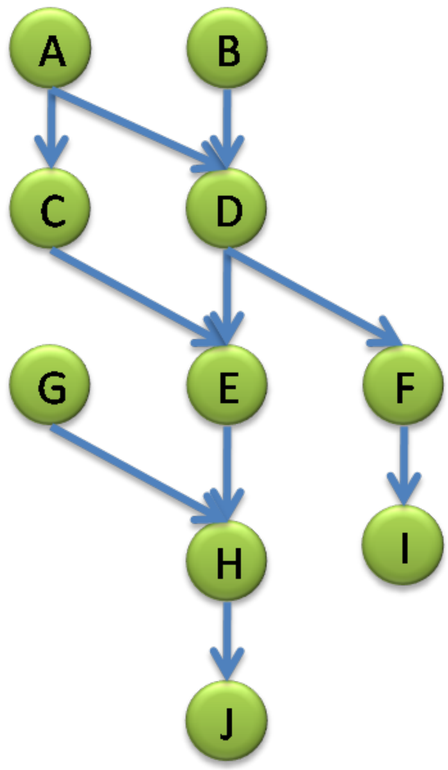
Therefore, $P(\neg a, b \mid d)$ is

$$\frac{P(\neg a)P(b \mid \neg a)P(d \mid b)}{\sum_b P(d \mid b) \sum_a P(a)P(b \mid a)}$$

, and in terms of (a) - (k),

$$\frac{(1-a) \cdot c \cdot f}{f(b \cdot a + c \cdot (1-a)) + g((1-b) \cdot a + (1-c)(1-a))}$$

3. [15 pts.] Given the Bayesian Network



(a) [5/15 pts.] Which nodes constitute the Markov blanket of node E? To which nodes E is conditionally independent given E's Markov blanket?

(Solution) A vairable is conditionally independent of all other nodes in the network given its parents, children, and children's paretns (co-parents). This is its Markov blanket.

Markov blanket: C,D,G,H

Conditionally independent nodes given the Markov blanket: A,B,F,I,J

(b) [10/15 pts.] For each of the following conditional independences, decide whether True or False. ($I(A; B \mid C)$ is True if A and B are conditionally independent given C. Otherwise, it is False.)

(Solution) You need to use d-separation to decide the following conditional independencies. A set of Variables X is conditionally independent of a set of Variables Y given a set of Evidence Variables E if all paths connecting an x to y are "d-separated." For more information, please refer to the following link:
`http://en.wikipedia.org/wiki/Bayesian_network#d-separation`

i. $I(E; J \mid D)$ False
ii. $I(E; I \mid D)$ True
iii. $I(E; A \mid D)$ False
iv. $I(G; I \mid E)$ True
v. $I(A; F \mid B)$ False