Probability Refresher and Cycle Analysis

A Quick Probability Refresher

- A random variable, X, can take on a number of different possible values
  - Example: the number of pigeons on the windowsill outside is a random variable with possible values 1,2,3,…
- Each time we observe (or sample) the random variable, it may take on a different value

A Quick Probability Refresher

- A random variable takes on each of these values with a specified probability
  - Example: $X = \{0, 1, 2, 3, 4\}$
  - $P[X=0] = .1$, $P[X=1] = .2$, $P[X=2] = .4$, $P[X=3] = .1$, $P[X=4] = .2$
- The sum of the probabilities of all values equals 1
  - $\Sigma_{all\ values} P[X=value] = 1$

A Quick Probability Refresher

- Example
  - Suppose we throw two dice and the random variable, X, is the sum of the two dice
  - Possible values of X are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
  - $P[X=2] = P[X=12] = 1/36$
  - $P[X=3] = P[X=11] = 2/36$
  - $P[X=4] = P[X=10] = 3/36$
  - $P[X=5] = P[X=9] = 4/36$
  - $P[X=6] = P[X=8] = 5/36$
  - $P[X=7] = 6/36$

Note: $\sum_{i=2}^{12} P[X=i] = 1$

A Quick Probability Refresher

- Expected Value
  - Can be thought of a "long term average" of observing the random variable a large number of times

$$E[X] = \overline{x} = \sum_{\substack{\text{All possible} \\ \text{values of } x}} Value * P[X = value]$$

- Example: dice - *E[X]*

  = $2*1/36 + 3*2/36 + 4*3/36 + 5*4/36 + 6*5/36 + 7*6/36 + 8*5/36 + 9*4/36 + 10*3/36 + 11*2/36 + 12*1/36$

Probability Example

- Basic probability notions
  - Two useful rules
    - Probabilities of all possible events sum to 1
    - Probability of independent events
      - Product of probabilities of events
      - e.g., probability of two coins coming up heads $= 1/2 \times 1/2 = 1/4$
  - Calculating averages/expected values
    - Function *f*
    - Multiply *f* by probability for each possible event
    - Sum over all events

Probability Example - Problem

- Given a bag with *N* balls
  - *1 blue* ball
  - *N - 1 white* balls
- Algorithm
  - pick a ball
    - if *blue*, you win
    - else return to bag
  - repeat *N* times
- Question
  - What is your chance of winning for large *N*?

Probability Example - Solution

- Can write as a sum
  - Chance of finding *blue* on first try $= 1/N$
  - On second try $= [(N-1)/N] * (1/N)$
  - Etc.
- Instead, write
  - $1 - (\text{chance of losing})$
  - Parenthesized term
    - Product of N factors
    - Each factor $= (N-1)/N$
  - $1 - [(N - 1)/N]^N$

Probability Example - Solution

- For $N = 2$,
  - $1/2$ first is *white*
  - $1/2$ second is *white*
  - $1/4$ both are *white*
  - $3/4$ chance to win $= 1 - (1/2)^2$
- For $N=3$,
  - $2/3$ first is *white*
  - $2/3$ second is *white*
  - $2/3$ third is *white*
  - $8/27$ all three are *white*
  - $19/27$ chance to win $= 1 - (2/3)^3 \quad (< 3/4)$

Probability Example - Solution

- *N=4* probability of win = 68%
- *N=5* probability of win = 67%
- *N=8* probability of win = 66%
- large *N*? *0*?

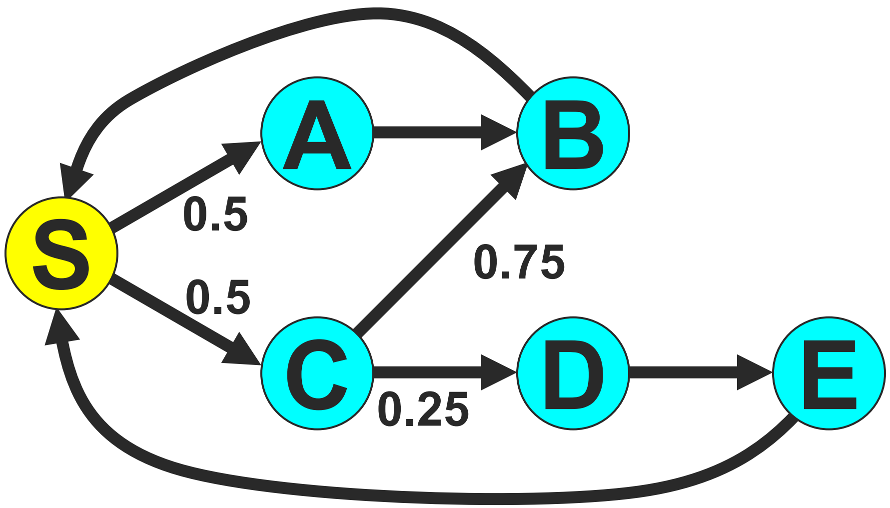
$$\lim_{N\to\infty}\left(\frac{N-1}{N}\right)^N$$

Fun Example

- Flip a coin repeatedly.
  - Two heads in a row scores 1 point.
  - Scoring pairs may not overlap
    - (e.g., three heads in a row does not score 2 points).
- On average, how many points do you score per flip?

A Different Example

- What fraction of time (on average) is spent in state E?

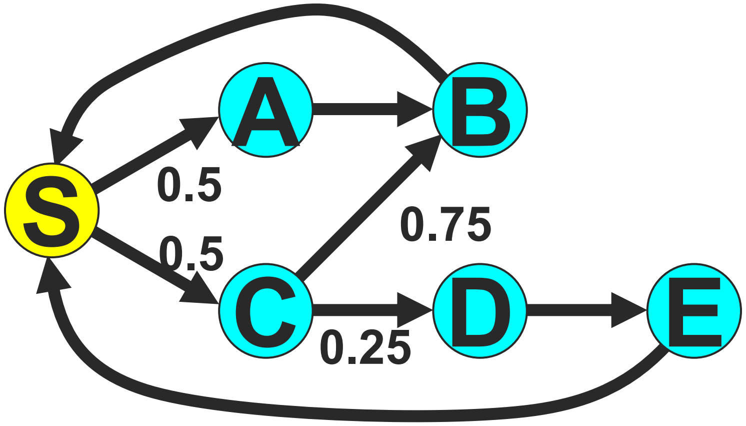


Cycle Analysis

- Start with a discrete Markov process
  - Transitions happen periodically (every Δt)
  - Probabilities independent of past/future behavior
- Form all possible cyclic sequences (cycles)
  - Pick a “start” state
  - List all cycles from that state
  - Calculate probability per cycle
  - Calculate average cycle length
- Can calculate expected values of cycle-dependent properties with average length and cycle probabilities

Example

cycle **probability**



Example

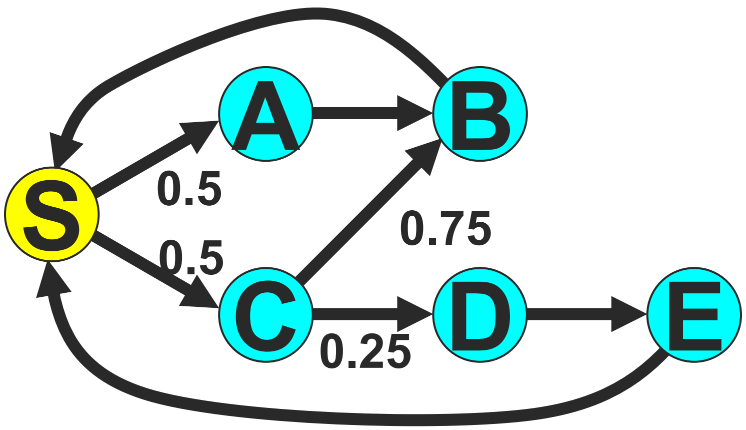
S→A A→B B→S

cycle **probability**

ABS

CBS

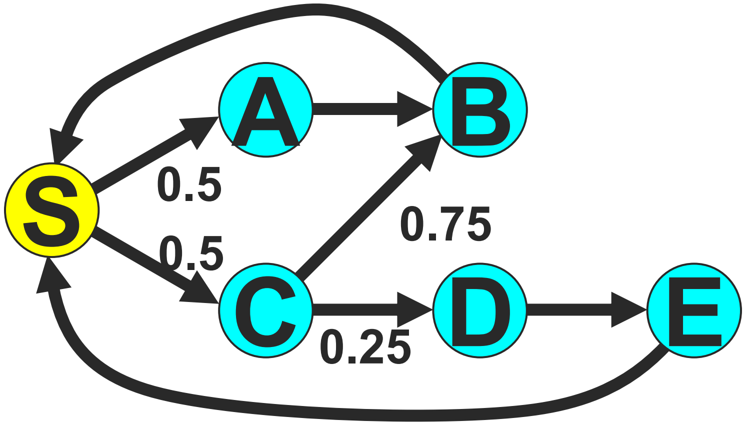
CDES



average cycle length

ABS CBS CDES

Example



Amount of time spent in E when in cycle CDES

- average fraction of time spent in E
  = 1•0.125 periods/cycle

Probability of cycle CDES

- dividing by average length…
  = 0.125 / 3.125 = 0.04

Fun Example

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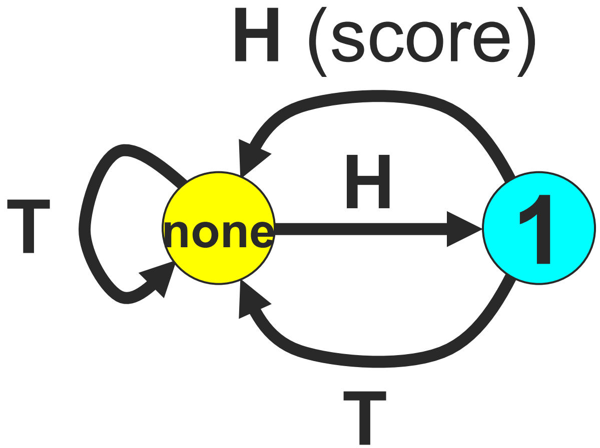
Fun Example

cycle	probability
T	1/2
HT	1/4
HH	1/4



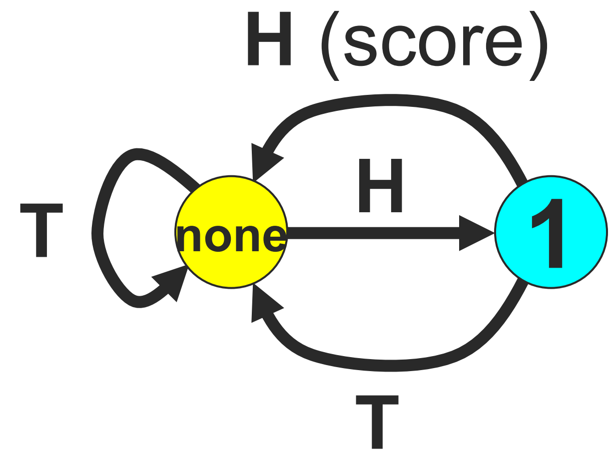
average cycle length

average score per cycle

average score per flip

Fun Example

- cycle probability
- T 1/2
- HT 1/4
- HH 1/4



average cycle length = 1/2 + 1/2 + 1/2 = 3/2 flips
average score per cycle = 1/4 points
average score per flip = (1/4) / (3/2) = 1/6 pts/flip