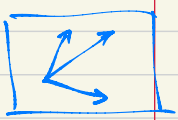
ECE/CS 434: Lin. Alg. Basics (3/2)

- **Basis**: Linearly Independent vectors that span a space

$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- **Dimensions** = # of vectors in basis = | Basis |

  ↳ A space can have multiple bases but all such bases have equal dim.



Q: $\dim(C(A)) = ?$ Rank of $A = \dim(R(A))$

- **Orthogonal vectors**: $\bar{x} \cdot \bar{y} = x^T y = 0$

  Then x and y are orthogonal

Q: Wall and floor orthogonal?

Ans: No. The intersection line have vectors $\in$ Wall and $\in$ floor but are not orthogonal.

- **Norm**: $\underset{m}{\begin{bmatrix} | \\ v_1 \\ | \end{bmatrix}} \longrightarrow$ scalar $\Rightarrow \mathbb{R}^m \to \mathbb{R}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

L_1 norm $= (x_1' + x_2' + x_3')'$
$= x_1 + x_2 + x_3$

L_2 norm $= \underbrace{\sqrt{x_1^2 + x_2^2 + x_3^2}}_{\text{mag. or length}}$

L_∞ norm = max value.

$$\|\bar{v}\|_p = \underbrace{\left(\sum_i |x_i|^p\right)^{1/p}}_{\text{scalar}} \quad \ldots \text{ called p-norm}$$

- **Length** = L_2 norm $= \sqrt{v_1^2 + v_2^2 + \ldots v_m^2} = \sqrt{v^T v}$

- **L_0 norm** = # of non-zero elements in the vector.

- **uv^T** = $\begin{bmatrix} | \\ | \end{bmatrix} \begin{bmatrix} \text{———} \end{bmatrix}$ = Rank 1 matrix

$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$

$v_1 v_1 + v_2 v_2 = 0$

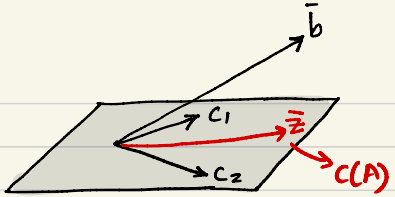
$\begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} = 0$

- **Symmetric matrix** $\equiv A = A^T$
- **$(AB)^{-1}$** $= B^{-1}A^{-1}$ and $(AB)^T = B^T A^T$

- **$A^T A$** is a square symm. matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ca+db & c^2+d^2 \end{bmatrix}$

$A = \underset{m}{\begin{bmatrix} \ \end{bmatrix}}^{n}$ $\quad \underset{n}{[\quad\quad]}^{m} \underset{m}{\begin{bmatrix} \ \end{bmatrix}}^{n} = n \times n$ matrix

$$A\bar{x} = \bar{b}$$



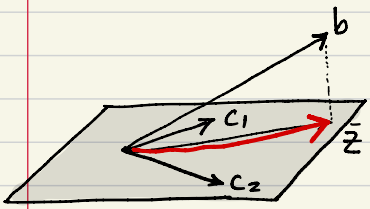
Q: But say you have to solve $A\bar{x} = \bar{b}$ even though $\bar{b} \notin C(A)$. What's the best you can do?

Ans: Produce a vector as $\bar{z} = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

s.t. $\bar{z}$ is closest possible to $\bar{b}$.

Now declare $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as the approx soln. to $A\bar{x} = \bar{b}$

What is "closest possible"?



Define closest possible as min length of the vector $(\bar{b} - \bar{z})$ $\Rightarrow$ or least square error

i.e., $\bar{z}$ is projection of $\bar{b}$ on C(A)

Given this, find solution to $\bar{x}$ which is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

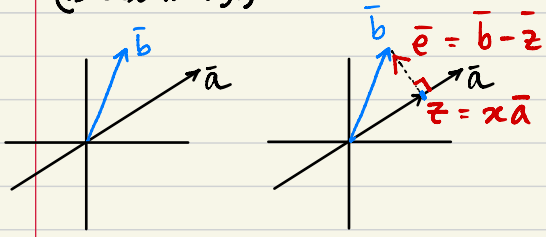
$$\begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} | \\ z \\ | \end{bmatrix}$$

$$Ax = b$$

$$\begin{bmatrix} | \\ a \\ | \end{bmatrix} [x] = \begin{bmatrix} | \\ b \\ | \end{bmatrix}$$

⊕ Least Squares Solution

(1D case w.l.o.g.)



$$\bar{z} = x \cdot \bar{a}$$

$$\bar{z} + \bar{e} = \bar{b}$$

$$\bar{e} = \bar{b} - \bar{z} = \bar{b} - x\bar{a}$$

Now, since $\bar{a} \perp \bar{e}$, we have $a^T \cdot e = 0$ $\quad e^T \cdot xa = 0$

or $\quad a^T(\bar{b} - x\bar{a}) = 0$ $\quad$ or $\quad a^T b - a^T x a = 0$

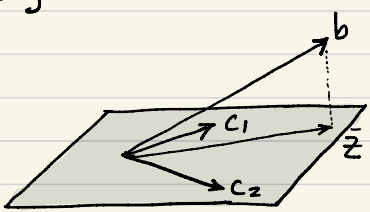
$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\therefore \quad x = \frac{a^T b}{a^T a}$$

solved $Ax = \hat{b}$ where $\hat{b}$ is the closest L_2 norm approx. of original $\bar{b}$.

⊕ Now, take this to higher dimensions

say 2D.



$$A\bar{x} = \bar{b}$$

$$\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}_{m \times n} \begin{bmatrix} | \\ x \\ | \end{bmatrix}_{n \times 1} = \begin{bmatrix} | \\ b \\ | \end{bmatrix}_{m \times 1}$$

$$A^T b = \begin{bmatrix} \cdots \\ \cdots \end{bmatrix}_{n \times m} \begin{bmatrix} | \\ b \\ | \end{bmatrix}_{m \times 1}$$

$A^T b$ = $n \times 1$ vector

$$\underbrace{\bar{x}}_{n \times 1 \text{ vector}} = \frac{A^T b}{(A^T A)} = (A^T A)^{-1} A^T b$$

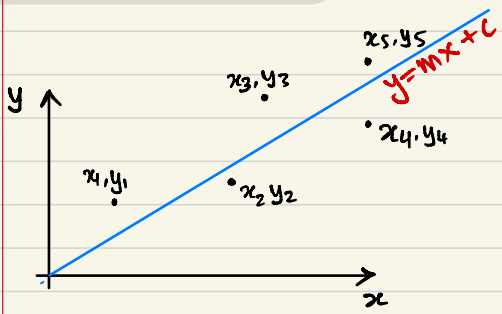
Projected vector:

$$\bar{z} = A\bar{x} = \frac{A A^T b}{(A^T A)}$$

$$\therefore \quad \text{Projection matrix } (P) = (A^T A)^{-1} A A^T$$

Matrix P projects any vector to the column space of A.

Example : Regression ⇒ Which line $y = mx + c$ satifies all points.

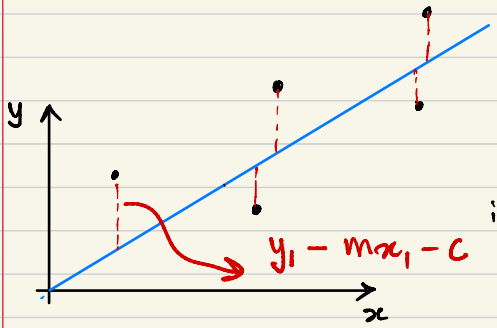


$$y_1 = mx_1 + c$$
$$y_2 = mx_2 + c$$
$$\vdots$$
$$y_n = mx_n + c$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\bar{b}}$$

Since no solution exists $\Rightarrow \bar{b} \notin C(A)$

so minimize error $e_i = \bar{b} - A\bar{x} = y_i - (mx_i + c)$



For this minimization, we know solution $\bar{x}$

i.e., $\bar{x} = \dfrac{A^T b}{(A^T A)}$

$$\begin{bmatrix} m \\ c \end{bmatrix} = \left(\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Eigenvalues and Eigenvectors.

$$A\bar{x} = b$$

A: function or operation; $\bar{x}$: input; b: output

or $\bar{x} \rightarrow \boxed{A(\cdot)} \rightarrow \bar{b}$

$$A\bar{x} = \lambda\bar{x}$$

A: square matrix

input vector only changes in mag. but not in direction ...

$x \equiv$ eigenvectors of A, $x \neq 0$

$\lambda \equiv$ eigenvalue of A

How to obtain eigenvectors and eigenvalues?

$$Ax = \lambda x \Rightarrow A\bar{x} - \lambda\bar{x} = 0 \Rightarrow (A - \lambda I)x = 0, \; x \neq 0$$

Null space exists for $(A - \lambda I)$

Thus, $\text{Det}(A - \lambda I) = 0$

e.g., $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\therefore (A - \lambda I) = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

$$\text{Det}(A - \lambda I) = (3-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow (\lambda - 4)(\lambda - 2) = 0$$

Characteristic polynomial

Trace $= \sum_i \lambda_i$

Det(A) $= \prod_i \lambda_i$

$$\therefore \lambda_1 = 4, \quad \lambda_2 = 2$$

Now, $\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigenvector #1

eigenvector #2

Eigen Decomposition and Diagonalization

$$\left.\begin{aligned} Ax_1 &= \lambda_1 x_1 \\ Ax_2 &= \lambda_2 x_2 \\ &\vdots \\ Ax_n &= \lambda_n x_n \end{aligned}\right\} \quad A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \\ \vdots & & 0 & \lambda_n \end{bmatrix}$$

A $S \rightarrow$ matrix of eigenvectors in order $= S\Lambda$ (S: matrix, Λ: matrix)

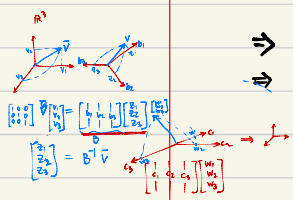
$$\therefore \quad AS = S\Lambda$$

$$AS = S\Lambda$$

$$\Rightarrow A = S\Lambda S^{-1} \rightarrow \text{called Eigen decomposition}$$

$$\Rightarrow S^{-1}AS = \Lambda \rightarrow \text{called Diagonalization}$$

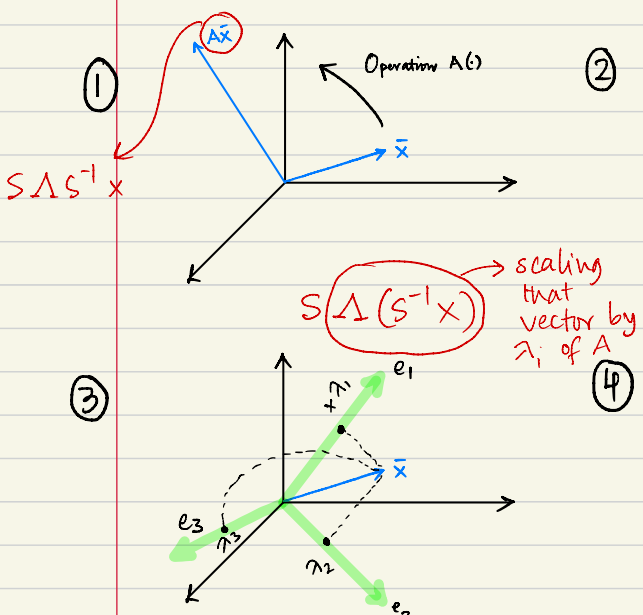
(Λ: matrix)



$A\bar{x} \equiv S\Lambda S^{-1}x$

$S\Lambda S^{-1}x$: $S^{-1}x$ → expressing $\bar{x}$ in Evector basis

Intuition for $A = S\Lambda S^{-1}$



So what is A^2 ? A^3 ? A^n ?

$A^2x = A(Ax)$ $\quad Ax = S\Lambda^2 S^{-1}x$ $\quad A^n x = S\Lambda^n S^{-1}x$

A ✓ $\longrightarrow$ Change of basis from I to E $\rightarrow$ Scaling ✓ $\longrightarrow$ Change back of basis from E to I