Problem 1: State True/False with a 1 line justification [5x4=20 points]

(a) A is a \( m \times n \) matrix with \( m < n \). The null space \( N(A) \) is always 0.

(b) The matrix \( A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), when left multiplied with a second matrix \( B \) (i.e. \( B \ast A \)), subtracts twice of the first column of \( B \) from the second column of \( B \).

(c) For an orthonormal matrix \( Q \) (i.e., columns of \( Q \) are orthogonal to each other and the length of each column is 1), \( Q^{-1} = Q^T \).

(d) If \( b_1, b_2, b_3 \) form the basis of a space, then \( c_1b_1 + c_2b_2 + c_3b_3 = 0 \) implies that all \( c_1 = c_2 = c_3 = 0 \).

(e) Matrix \( A \) has 6 columns, each column being a 10 dimensional vector. You are told that \( N(A^T) \) is 5. Then, \( N(A) \) must be 1 and Rank(A) must be 5.

Problem 2: Symmetric Matrices [10+10=20 points]

(a) Prove that \( A^T A \) is a symmetric matrix.

Hint: use the basic properties of transpose, as discussed in class.

(b) Prove that \( \text{Rank}(AB) \leq \min\{ \text{Rank}(A), \text{Rank}(B) \} \)

Problem 3: Column Spaces [10+10=20 points]

(a) Choose \( b \) which gives no solution and another \( b \) which gives infinitely many solutions. Your answer should show two values of \( b \).

\[
3x + 2y = 10 \\
6x + 4y = b
\]

(b) Consider matrix \( A_{m \times n} \). You are told \( r = \text{Rank}(A) \) and \( r < m \) and \( r < n \). How many solutions are possible for the equation \( Ax = b \)? What is the dimensions of \( N(A) \)?
Problem 4 : Least Squares

Consider the following system of equations (called an over-determined system since there are more equations than unknowns):

\[ x - y = 2 \] \hspace{1cm} (3)
\[ x + y = 4 \] \hspace{1cm} (4)
\[ 2x + y = 8 \] \hspace{1cm} (5)

How many solutions exist for the above system of equations? If a solution exists find one, if not, determine the least squares solution for \( x \) and \( y \).

Problem 5 : Eigen values and Eigen vectors \hspace{1cm} \[10+10=20\text{ points}\]

(a) Prove that, for symmetric matrix \( A \), eigenvalues of matrix \( A^2 \) = (Eigenvalue of matrix \( A \))^2

(b) Prove that \[ \lambda \left( A - \sigma I \right) = \left( \lambda(A) - \sigma \right) \]