

Lambda Calculus

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Objectives

You should be able to...

The purposes of this lecture is to introduce lambda calculus and explain the role it has in programming languages.

- ▶ Explain the three constructs of λ -calculus.
- ▶ Explain why on earth anybody would think this to be a good idea
- ▶ Contrast Lambda calculus with the von Neumann architecture as a foundation of a programming language.
- ▶ Show how to perform a beta-reduction
- ▶ Explain alpha-capture and some techniques to avoid it
- ▶ Show how this can be implemented in Haskell.

The λ -Calculus

- ▶ Contains three kinds of things:
 1. Variables: x — assume they are all one letter long.
 2. Functions: $\lambda x.M$ (Also called *abstractions*.)
 - ▶ The λ indicates the start of a function.
 - ▶ The x is the parameter.
 - ▶ The $.$ separates the parameter list from the function body.
 - ▶ The M is the body of the function. (Note: not a literal M .)
 3. Application: fy
- ▶ “We don’t need no stinking integers!”
- ▶ This language is Turing Complete.
- ▶ Used extensively in research. The “little white mouse” of computer science.
- ▶ We can implement this trivially in Haskell.
 $\lambda x.x = \backslash x \rightarrow x.$

Examples of notation

- ▶ To write down a function, write a λ , then the name of the variable to be bound (*parameter*), a dot, and the body. The body goes “as far as it can.”
- ▶ To write down a function application, use juxtaposition (like Haskell).

$$\lambda x.x$$

$$\lambda y.yx$$

$$\lambda z.\lambda x.x(zx)$$

- ▶ You can combine lambdas together.

$$\lambda z.\lambda x.x(zx) == \lambda zx.x(zx)$$

Syntax Trees

Example 1

Example 2

Example 3

Example 4

$$\lambda x$$

$$\downarrow$$

$$x$$
 $\lambda x.x$

$$\lambda y$$

$$\downarrow$$

$$@$$

$$\swarrow \searrow$$

$$y \quad x$$
 $\lambda y.yx$

$$@$$

$$\swarrow \searrow$$

$$\lambda y \quad x$$

$$\downarrow$$

$$y$$
 $(\lambda y.y)x$

$$\lambda z$$

$$\downarrow$$

$$@$$

$$\swarrow \searrow$$

$$\lambda x \quad \lambda y$$

$$\downarrow \quad \downarrow$$

$$@ \quad @$$

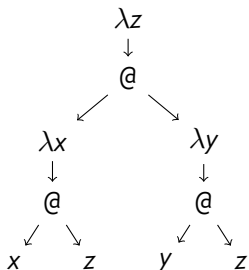
$$\swarrow \searrow \quad \swarrow \searrow$$

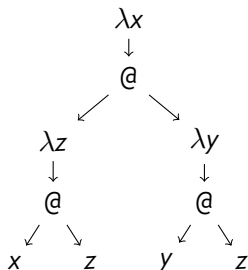
$$x \quad z \quad y \quad z$$
 $\lambda z.(\lambda x.xz)(\lambda y.yz)$

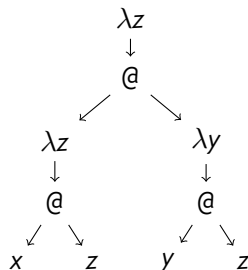
Bound and Free

- ▶ The λ creates a *binding*.
- ▶ An occurrence of the the variable inside the function body is said to be *bound*.
- ▶ Bound variables occur “under the λ ” that binds them.

Examples: Where are the free variables? To which lambdas are bound variables bound?



$$\lambda z.(\lambda x.xz)(\lambda y.yz)$$


$$\lambda x.(\lambda z.xz)(\lambda y.yz)$$


$$\lambda z.(\lambda z.xz)(\lambda y.yz)$$

Function application

$$(\lambda x.M)N \mapsto [N/x]M$$

$$[N/x] y = y$$

$$[N/x] x = N$$

$$[N/x] (M P) = ([N/x]M [N/x]P)$$

$$[N/x] (\lambda y.M) = \lambda y.[N/x]M$$

$$[N/x] (\lambda x.M) = \lambda x.M$$

Examples

 $(\lambda x.x) a$ $(\lambda x.x x) a$ $(\lambda x.y x) a$ $(\lambda x.\lambda a.x) a$ $(\lambda x.\lambda x.x) a$ $(\lambda x.(\lambda y.y) x) a$

Examples

$$(\lambda x.x) a \rightarrow_{\beta} a$$

$$(\lambda x.x x) a$$

$$(\lambda x.y x) a$$

$$(\lambda x.\lambda a.x) a$$

$$(\lambda x.\lambda x.x) a$$

$$(\lambda x.(\lambda y.y) x) a$$

$$\begin{array}{ccc} & @ & \rightarrow_{\beta} a \\ \swarrow & & \searrow \\ \lambda x & & a \\ \downarrow & & \\ x & & \end{array}$$

Examples

$$(\lambda x.x) a \rightarrow_{\beta} a$$

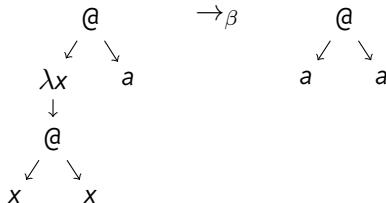
$$(\lambda x.x x) a \rightarrow_{\beta} a a$$

$$(\lambda x.y x) a$$

$$(\lambda x.\lambda a.x) a$$

$$(\lambda x.\lambda x.x) a$$

$$(\lambda x.(\lambda y.y) x) a$$



Examples

$$(\lambda x.x) a \rightarrow_{\beta} a$$

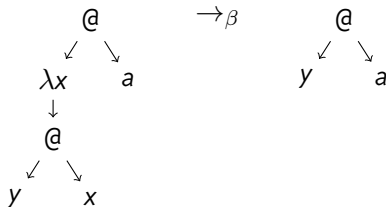
$$(\lambda x.x x) a \rightarrow_{\beta} a a$$

$$(\lambda x.y x) a \rightarrow_{\beta} y a$$

$$(\lambda x.\lambda a.x) a$$

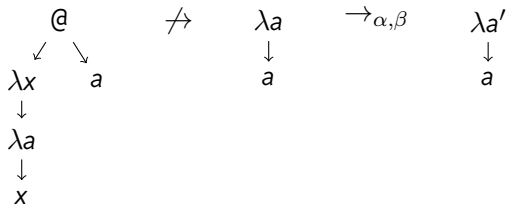
$$(\lambda x.\lambda x.x) a$$

$$(\lambda x.(\lambda y.y) x) a$$



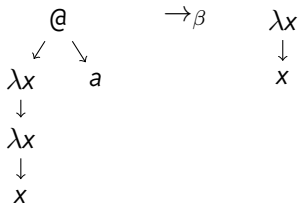
Examples

$(\lambda x.x) a$	$\rightarrow_{\beta}$	a	
$(\lambda x.x x) a$	$\rightarrow_{\beta}$	$a a$	
$(\lambda x.y x) a$	$\rightarrow_{\beta}$	$y a$	
$(\lambda x.\lambda a.x) a$	$\rightarrow_{\alpha}$	$(\lambda x.\lambda a'.x)$	$\rightarrow_{\beta}$ $\lambda a'.a$
$(\lambda x.\lambda x.x) a$			
$(\lambda x.(\lambda y.y) x) a$			



Examples

$(\lambda x.x) a$	$\rightarrow_{\beta}$	a	
$(\lambda x.x x) a$	$\rightarrow_{\beta}$	$a a$	
$(\lambda x.y x) a$	$\rightarrow_{\beta}$	$y a$	
$(\lambda x.\lambda a.x) a$	$\rightarrow_{\alpha}$	$(\lambda x.\lambda a'.x)$	$\rightarrow_{\beta}$ $\lambda a'.a$
$(\lambda x.\lambda x.x) a$	$\rightarrow_{\beta}$	$\lambda x.x$	
$(\lambda x.(\lambda y.y) x) a$			



Examples

$$(\lambda x.x) a \rightarrow_{\beta} a$$

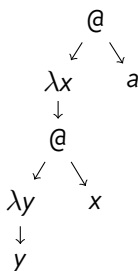
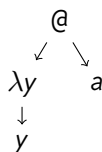
$$(\lambda x.x x) a \rightarrow_{\beta} a a$$

$$(\lambda x.y x) a \rightarrow_{\beta} y a$$

$$(\lambda x.\lambda a.x) a \rightarrow_{\alpha} (\lambda x.\lambda a'.x) \rightarrow_{\beta} \lambda a'.a$$

$$(\lambda x.\lambda x.x) a \rightarrow_{\beta} \lambda x.x$$

$$(\lambda x.(\lambda y.y) x) a \rightarrow_{\beta} (\lambda y.y) a$$


 $\rightarrow_{\beta}$

 $\rightarrow_{\beta}$
 a

Examples

$$(\lambda x.x) a \rightarrow_{\beta} a$$

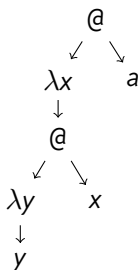
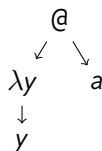
$$(\lambda x.x x) a \rightarrow_{\beta} a a$$

$$(\lambda x.y x) a \rightarrow_{\beta} y a$$

$$(\lambda x.\lambda a.x) a \rightarrow_{\alpha} (\lambda x.\lambda a'.x) \rightarrow_{\beta} \lambda a'.a$$

$$(\lambda x.\lambda x.x) a \rightarrow_{\beta} \lambda x.x$$

$$(\lambda x.(\lambda y.y) x) a \rightarrow_{\beta} (\lambda y.y) a \rightarrow_{\beta} a$$


 $\rightarrow_{\beta}$

 $\rightarrow_{\beta}$
 a

Large Example

$$\begin{aligned}
 &(\lambda a_b. \lambda a_p. \\
 &\quad (a_p (a_b \lambda x. x) \\
 &\quad\quad (a_b \lambda y. y))) \\
 &(\lambda e. e) \\
 &(\lambda e_1. \lambda e_2. (e_1 e_2)))
 \end{aligned}$$

Here is a larger example. It is a lambda calculus interpreter written in lambda calculus, applied to the expression

$$(\lambda x. x)(\lambda y. y)$$

Don't worry....

Large Example, pt 2

$$\begin{aligned}
 &(\lambda a_b. \lambda a_p. \\
 &\quad (a_p(a_b \lambda x. x) \\
 &\quad\quad (a_b \lambda y. y))) \\
 &(\lambda e. e) \\
 &(\lambda e_1. \lambda e_2. (e_1 e_2)))
 \end{aligned}$$

Large Example, pt 2

$$\begin{aligned}
 &(\lambda a_b. \lambda a_p. \\
 &\quad (a_p(a_b \lambda x.x) \\
 &\quad\quad (a_b \lambda y.y))) \\
 &(\lambda e.e) \\
 &(\lambda e_1. \lambda e_2. (e_1 e_2)))
 \end{aligned}$$

becomes...

$$\begin{aligned}
 &(\lambda a_p. \\
 &\quad (a_p((\lambda e.e) \lambda x.x) \\
 &\quad\quad ((\lambda e.e) \lambda y.y))) \\
 &(\lambda e_1. \lambda e_2. (e_1 e_2)))
 \end{aligned}$$

Large Example, pt 3

$$\begin{aligned} &(\lambda a_p. \\ &\quad (a_p((\lambda e.e)\lambda x.x) \\ &\quad\quad ((\lambda e.e)\lambda y.y)) \\ &(\lambda e_1.\lambda e_2.(e_1 e_2))) \end{aligned}$$

Large Example, pt 3

$$\begin{aligned}
 &(\lambda a_p. \\
 &\quad (a_p((\lambda e.e)\lambda x.x) \\
 &\quad\quad ((\lambda e.e)\lambda y.y)) \\
 &(\lambda e_1.\lambda e_2.(e_1 e_2)))
 \end{aligned}$$

becomes...

$$\begin{aligned}
 &((\lambda e_1.\lambda e_2.(e_1 e_2)) \\
 &\quad ((\lambda e.e)\lambda x.x) \\
 &\quad ((\lambda e.e)\lambda y.y))
 \end{aligned}$$

Large Example, pt 4

$$\begin{aligned} & ((\lambda e_1. \lambda e_2. (e_1 e_2)) \\ & \quad ((\lambda e. e) \lambda x. x) \\ & \quad ((\lambda e. e) \lambda y. y)) \end{aligned}$$

Large Example, pt 4

$$\begin{aligned}
 &((\lambda e_1. \lambda e_2. (e_1 e_2)) \\
 &\quad ((\lambda e. e) \lambda x. x) \\
 &\quad ((\lambda e. e) \lambda y. y))
 \end{aligned}$$

becomes...

$$\begin{aligned}
 &(\lambda e_2. (((\lambda e. e) \lambda x. x) e_2)) \\
 &\quad ((\lambda e. e) \lambda y. y)
 \end{aligned}$$

Large Example, pt 5

$$(\lambda e_2. (((\lambda e.e)\lambda x.x)e_2)) \\ ((\lambda e.e)\lambda y.y)$$

becomes...

$$(((\lambda e.e)\lambda x.x)((\lambda e.e)\lambda y.y))$$

Large Example, pt 6

$$(((\lambda e.e)\lambda x.x)((\lambda e.e)\lambda y.y))$$

Large Example, pt 6

$$(((\lambda e.e)\lambda x.x)((\lambda e.e)\lambda y.y))$$

becomes...

$$(\lambda x.x)((\lambda e.e)\lambda y.y)$$

Large Example, pt 6

$(((\lambda e.e)\lambda x.x)((\lambda e.e)\lambda y.y))$

becomes...

$(\lambda x.x)((\lambda e.e)\lambda y.y)$

Large Example, pt 6

$$(((\lambda e.e)\lambda x.x)((\lambda e.e)\lambda y.y))$$

becomes...

$$(\lambda x.x)((\lambda e.e)\lambda y.y)$$

becomes...

$$(\lambda e.e)\lambda y.y$$

Large Example, pt 6

$$(((\lambda e.e)\lambda x.x)((\lambda e.e)\lambda y.y))$$

becomes...

$$(\lambda x.x)((\lambda e.e)\lambda y.y)$$

becomes...

$$(\lambda e.e)\lambda y.y$$

Large Example, pt 6

$$(((\lambda e.e)\lambda x.x)((\lambda e.e)\lambda y.y))$$

becomes...

$$(\lambda x.x)((\lambda e.e)\lambda y.y)$$

becomes...

$$(\lambda e.e)\lambda y.y$$

which finally becomes...

$$\lambda y.y$$