Hwk10 Solution: Polymorphic types; Hoare axioms
CS 421 – Spring 2011

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1. In the following, Γ_0 is the initial environment (as usual), Γ_1 is $\Gamma_0[f : \forall\alpha.(int \to \alpha) \to \alpha]$ and Γ_2 is $\Gamma_1[y : int]$.

   $\Gamma_0 \vdash$ let f=fun x -> x 0 in f (fun y -> (+) y 1) : int (let)
   - $\Gamma_0 \vdash$ fun x -> x 0 : $(int \to \alpha) \to \alpha$ (abs)
     - $\Gamma_0[x : int \to \alpha] \vdash$ x 0 : α (app)
       - $\Gamma_0[x : int \to \alpha] \vdash$ x : $int \to \alpha$ (var)
       - $\Gamma_0[x : int \to \alpha] \vdash$ 0 : int (var)
   - $\Gamma_1 \vdash$ f (fun y -> (+) y 1) : int (app)
     - $\Gamma_1 \vdash$ f : $(int \to int) \to int$ (since $(int \to int) \to int \leq \forall\alpha.(int \to \alpha) \to \alpha$) (var)
     - $\Gamma_1 \vdash$ fun y -> (+) y 1 : $int \to int$ (abs)
       - $\Gamma_2 \vdash$ (+) y 1 : int (app)
         - $\Gamma_2 \vdash$ (+) y : $int \to int$ (app)
           - $\Gamma_2 \vdash$ (+) : $int \to int \to int$ (var)
           - $\Gamma_2 \vdash$ y : int (var)
         - $\Gamma_2 \vdash$ 1 : int (var)

2. In the following, Γ_3 is additionally defined as $\Gamma_1[n : int]$.

   $\Gamma_0 \vdash$ let f=fun x -> x 0 in (f (fun y -> (+) y 1), f(fun n -> (::) n [])) : $int * int\ list$ (let)
   - $\Gamma_0 \vdash$ fun x -> x 0 : $(int \to \alpha) \to \alpha$ (abs)
     - $\Gamma_0[x : int \to \alpha] \vdash$ x 0: α (app)
       - $\Gamma_0[x : int \to \alpha] \vdash$ x: $int \to \alpha$ (var)
       - $\Gamma_0[x : int \to \alpha] \vdash$ 0: int (var)
   - $\Gamma_1 \vdash$ (f (fun y -> (+) y 1), f(fun n -> (::) n [])) : $int * int\ list$ (tuple)
     - $\Gamma_1 \vdash$ f (fun y -> (+) y 1) : int (app)
       - $\Gamma_1 \vdash$ f : $(int \to int) \to int$ (since $(int \to int) \to int \leq \forall\alpha.(int \to \alpha) \to \alpha$) (var)
       - $\Gamma_1 \vdash$ fun y -> (+) y 1 : $int \to int$ (abs)
         - $\Gamma_2 \vdash$ (+) y 1 : int (app)
           - $\Gamma_2 \vdash$ (+) y : $int \to int$ (app)
             - $\Gamma_2 \vdash$ (+) : $int \to int \to int$ (var)
             - $\Gamma_2 \vdash$ y : int (var)
           - $\Gamma_2 \vdash$ 1 : int (var)
     - $\Gamma_1 \vdash$ f (fun n -> (::) n []) : $int\ list$ (app)
       - $\Gamma_1 \vdash$ f : $(int \to int\ list) \to int\ list$

(since $(int \to int\ list) \to int\ list \ \leq\ \forall\alpha.(int \to \alpha) \to \alpha)$ (var)

$\Gamma_1 \vdash$ fun n -> (::) n [] : $int \to int\ list$ (abs)

$\Gamma_3 \vdash$ (::) n [] : $int\ list$ (app)

$\Gamma_3 \vdash$ (::) n : $int\ list \to int\ list$ (app)

$\Gamma_3 \vdash$ (::) : $int \to int\ list \to int\ list$ (var)

$\Gamma_3 \vdash$ n : int (var)

$\Gamma_3 \vdash$ [] : $int\ list$ (since $\Gamma_0 \vdash$ [] : $\forall\alpha.\alpha\ list$ and $int\ list\ \leq\ \forall\alpha.\alpha\ list)$ (var)

3. Pre-condition: $min = 0\ \&\ i = 1$

   Post-condition: $\forall 0 \leq i < |A|.A[min] \leq A[i]$

   ```
   while (i<n) { if (A[i] < A[min]) min = i;
                 i = i+1; }
   ```

   Invariant: $\forall 0 \leq j < i.A[min] \leq A[j]\ \&\ i \leq |A|$

4. Pre-condition: $i = 0\ \&\ n = |A|$

   Post-condition: $\forall 0 \leq j < |A| - 1.A[j] \leq A[j+1]\ \&\ A$ has the same elements as A_0

   ```
   while (i < n-1) {
      m = min(A,i); swap(A, i, m); i = i+1;
   }
   ```

   Invariant: $\forall 0 \leq j < i.A[j] \leq A[j+1]\ \&\ i < |A|\ \&\ A$ **has the same elements as** A_0

5. Pre-condition: $lo = 0\ \&\ hi = |A| - 1$

   Post-condition: $\forall 0 \leq i < lo.A[i] \leq x\ \&\ \forall lo \leq i < |A|.A[i] > x$

   ```
   while (lo<hi) {
      if (A[lo] > x) {
         swap(A, lo, hi); hi = hi-1;
      }
      else if (A[hi] <= x) {
         swap(A, lo, hi); lo = lo+1;
      }
      else { lo = lo+1; hi = hi-1; }
   }
   ```

   Invariant: $lo \leq hi + 1\ \&\ \forall 0 \leq i < lo.A[i] \leq x\ \&\ \forall hi < i < |A|.A[i] > x$

6. Pre-condition: $i = 0\ \&\ j = |A| - 1$

   Post-condition: $\forall 0 \leq i < |A|.A[i]\ =\ A_0[|A| - i - 1]$

   ```
   while (i < j) { swap(A, i, j);  i = i+1; j = j-1;}
   ```

   Invariant: $\forall 0 \leq k < i.(A[k]\ =\ A_0[|A| - k - 1]\ \&\ A[|A| - k - 1]\ =\ A_0[k])\ \&\ i \leq \lceil |A|/2 \rceil\ \&\ j \geq \lfloor |A|/2 \rfloor$