

1)

$$\begin{array}{c}
 \frac{}{\{x:\alpha \rightarrow \text{int}, y:\alpha\} \vdash x: \alpha \rightarrow \text{int}} \quad \frac{}{\{x:\alpha \rightarrow \text{int}, y:\alpha\} \vdash y: \alpha} \\
 \hline
 \{x:\alpha \rightarrow \text{int}, y:\alpha\} \vdash x(y): \text{int} \quad \frac{}{\{x:\alpha \rightarrow \text{int}, y:\alpha\} \vdash 1: \text{int}} \\
 \hline
 \{x:\alpha \rightarrow \text{int}, y:\alpha\} \vdash x(y)+1: \text{int} \\
 \hline
 \{x:\alpha \rightarrow \text{int}\} \vdash \text{fun } y \rightarrow x(y)+1: \alpha \rightarrow \text{int} \\
 \hline
 \frac{}{\emptyset \vdash \text{let rec } x = \text{fun } y \rightarrow x(y)+1 \text{ in } x(1) : \text{int}}
 \end{array}$$

2)

$$\begin{array}{c}
 \frac{}{\{x:\alpha\} \vdash x: \alpha} \quad \frac{}{\{f:\forall \alpha. \alpha \rightarrow \alpha\} \vdash f : (\text{int} \rightarrow \text{int}) \rightarrow (\text{int} \rightarrow \text{int})} \quad \frac{}{\{f:\forall \alpha. \alpha \rightarrow \alpha\} \vdash f : \text{int} \rightarrow \text{int}} \\
 \hline
 \frac{}{\emptyset \vdash \text{fun } x \rightarrow x : \alpha \rightarrow \alpha} \quad \frac{}{\{f:\forall \alpha. \alpha \rightarrow \alpha\} \vdash (f f): \text{int} \rightarrow \text{int}} \quad \frac{}{\{f:\forall \alpha. \alpha \rightarrow \alpha\} \vdash 1: \text{int}} \\
 \hline
 \frac{}{\emptyset \vdash \text{let } f = \text{fun } x \rightarrow x \text{ in } (f f)1 : \text{int}}
 \end{array}$$

3)

$$\frac{\frac{}{\{y: 3\}, y \Downarrow 3} \quad \frac{}{\{y: 3\}, 1 \Downarrow 1}}{\mathbf{C} = \{y: 3\}, y+1 \Downarrow 4}$$

$$\frac{\frac{}{\eta, f \Downarrow \langle \text{fun } y \rightarrow y+1, \emptyset \rangle} \quad \frac{}{\eta, 2 \Downarrow 2} \quad \frac{\frac{}{\{y: 2\}, y \Downarrow 2} \quad \frac{}{\{y: 2\}, 1 \Downarrow 1}}{\{y: 2\}, y+1 \Downarrow 3}}{\mathbf{B} = \eta, (f \ 2) \Downarrow 3}$$

$$\frac{\frac{}{\emptyset, (\text{fun } f \rightarrow f(f \ 2)) \Downarrow \langle \text{fun } f \rightarrow f(f \ 2), \emptyset \rangle} \quad \frac{}{\emptyset, (\text{fun } y \rightarrow y+1) \Downarrow \langle \text{fun } y \rightarrow y+1, \emptyset \rangle} \quad \frac{\frac{}{\eta, f \Downarrow \langle \text{fun } y \rightarrow y+1, \emptyset \rangle} \quad \mathbf{B} \quad \mathbf{C}}{\{f: \langle \text{fun } y \rightarrow y+1, \emptyset \rangle, f(f \ 2) \Downarrow 4}}{\emptyset, (\text{fun } f \rightarrow f(f \ 2))(\text{fun } y \rightarrow y+1) \Downarrow 4}$$

To reduce notational clutter, use η as an abbreviation for the environment $\{f: \langle \text{fun } y \rightarrow y+1, \emptyset \rangle\}$.

4)

$$\begin{array}{c}
 \frac{}{\{l \rightarrow 0\}, \{x:l\} \vdash x \Downarrow l, \{l \rightarrow 0\}} \quad \frac{}{\{l \rightarrow 0\}(l) = 0} \\
 \hline
 \frac{}{\{l \rightarrow 0\}, \{x:l\} \vdash !x \Downarrow 0, \{l \rightarrow 0\}} \quad \frac{}{\{l \rightarrow 0\}, \{x:l\} \vdash 1 \Downarrow 1, \{l \rightarrow 0\}} \\
 \hline
 \frac{\{l \rightarrow 0\}, \{x:l\} \vdash x \Downarrow l, \{l \rightarrow 0\} \quad \{l \rightarrow 0\}, \{x:l\} \vdash !x+1 \Downarrow 1, \{l \rightarrow 0\}}{\mathbf{D} = \{l \rightarrow 0\}, \{x:l\} \vdash x := !x+1 \Downarrow (), \{l \rightarrow 1\}}
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\emptyset, \emptyset \vdash (\text{fun } x \rightarrow x := !x+1) \Downarrow \langle (\text{fun } x \rightarrow x := !x+1), \emptyset \rangle, \emptyset} \quad \frac{}{\emptyset, \emptyset \vdash 0 \Downarrow 0, \emptyset} \\
 \hline
 \frac{\emptyset, \emptyset \vdash (\text{fun } x \rightarrow x := !x+1) \Downarrow \langle (\text{fun } x \rightarrow x := !x+1), \emptyset \rangle, \emptyset \quad \emptyset, \emptyset \vdash \text{ref } 0 \Downarrow l, \{l \rightarrow 0\}}{\emptyset, \emptyset \vdash (\text{fun } x \rightarrow x := !x+1)(\text{ref } 0) \Downarrow (), \{l \rightarrow 1\}} \quad \mathbf{D}
 \end{array}$$

5)

$$\frac{\sigma \vdash e_1 \ e_2 \Downarrow (\text{fun } x \rightarrow e), \sigma_1 \quad \sigma_1 \vdash e_2 \Downarrow v_2, \sigma_2 \quad \sigma_2 \vdash e[v_2/x] \Downarrow v, \sigma_3}{\sigma \vdash e_1 \ e_2 \Downarrow v, \sigma_3}$$