Programming Languages and Compilers (CS 421)

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https://courses.engr.illinois.edu/cs421/fa2024/CS421C

Based on slides by Elsa Gunter, which are based in part on previous slides by Mattox Beckman and updated by Vikram Adve and Gul Agha

Recursion over Recursive Data Types

```
# type exp = VarExp of string | ConstExp of const
   | BinOpAppExp of bin_op * exp * exp
   | FunExp of string * exp | AppExp of exp * exp
```

- How to count the number of variables in an exp?

```
# let rec varCnt exp =
   match exp with
      VarExp x ->
    | ConstExp c ->
    | BinOpAppExp (b, e1, e2) ->
    | FunExp (x,e) ->
    | AppExp (e1, e2) ->
```

Why can't you pass data constructors around like regular functions in OCaml?

From the horse's mouth, Xavier Leroy, in this mailing list message from 2001:

> The old Caml V3.1 implementation treated constructors as functions like SML. In Caml Light, I chose to drop this equivalence for several reasons:
>
> - Simplicity of the compiler. Internally, constructors are not functions, and a special case is needed to transform Succ into (fun x -> Succ x) when needed. This isn't hard, but remember that Caml Light was really a minimal, stripped-down version of Caml.
> - Constructors in Caml Light and OCaml really have an arity, e.g. C of int * int is really a constructor with two integer arguments, not a constructor taking one argument that is a pair. Hence, there would be two ways to map the constructor C to a function: fun (x,y) -> C(x,y) or fun x y -> C(x,y) The former is more natural if you come from an SML background (where constructors have 0 or 1 argument), but the latter fits better the Caml Light / OCaml execution model, which favors curried functions. By not treating constructors like functions, we avoid having to choose...
> - Code clarity. While using a constructor as a function is sometimes convenient, I would argue it is often hard to read. Writing "fun x -> Succ x" is more verbose, but easier to read, I think.

From: https://stackoverflow.com/questions/66833935/why-cant-you-pass-data-constructors-around-like-regular-functions-in-ocaml

Mutually Recursive Types

```
# type 'a tree =
         TreeLeaf of 'a
       | TreeNode of 'a treeList
and
    'a treeList =
         Last of 'a tree
       | More of ('a tree * 'a treeList);;
```

type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList)

Mutually Recursive Types

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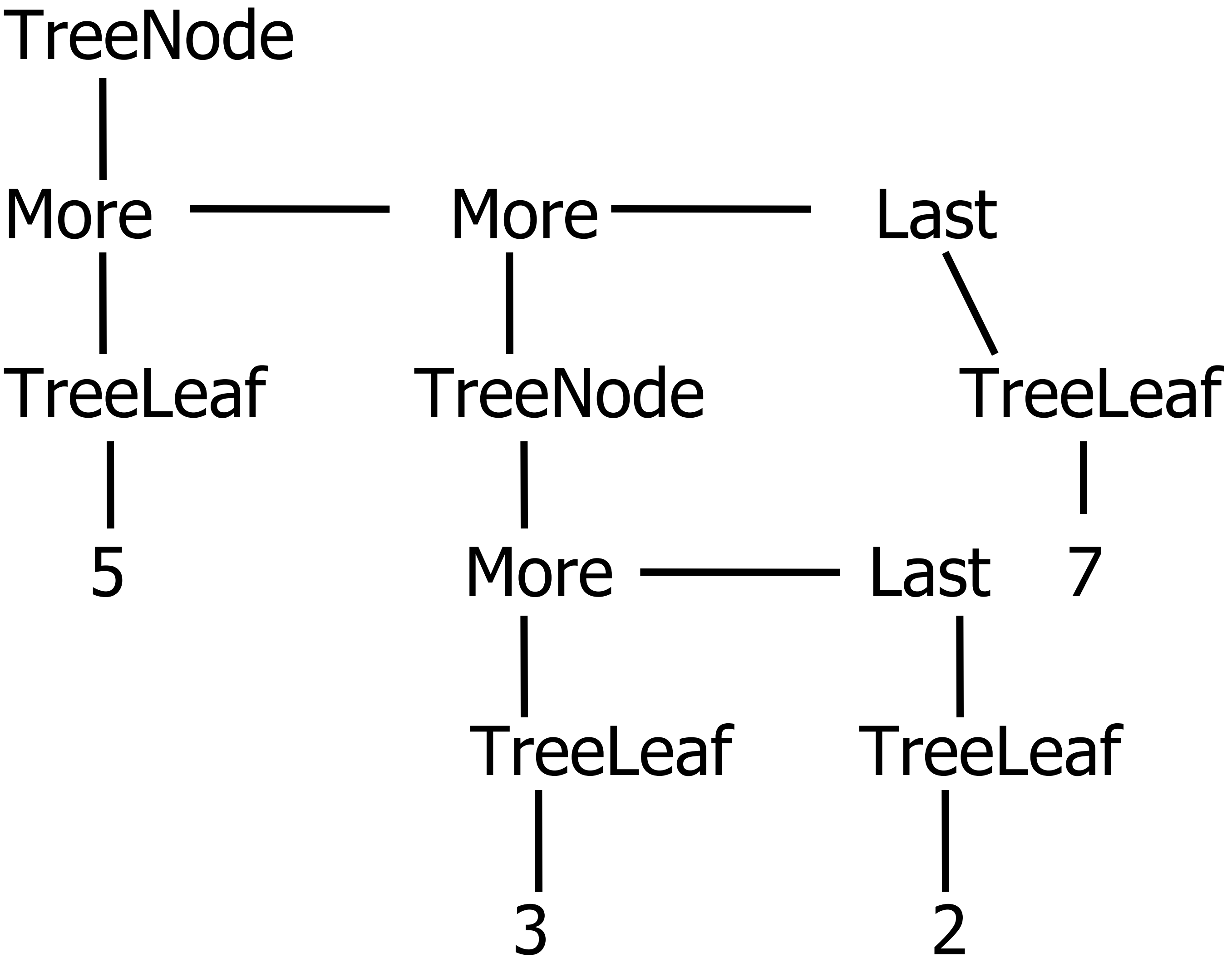
type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList

and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList)

Mutually Recursive Types - Values

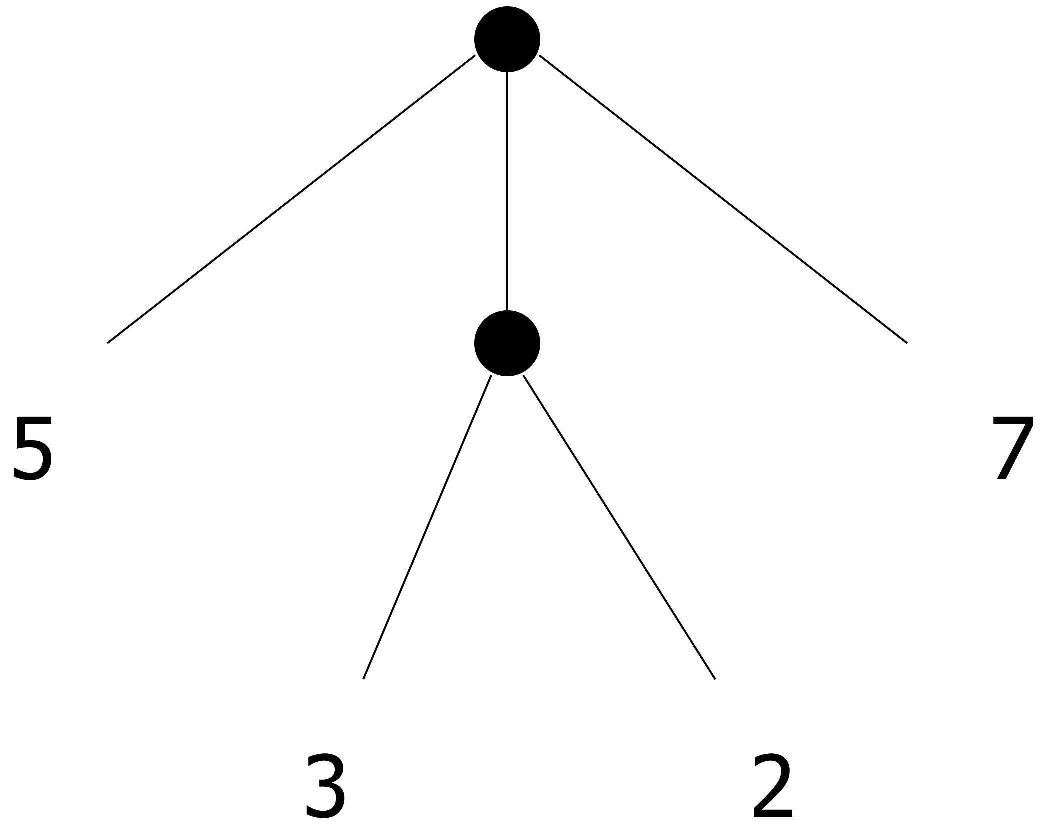
```
# let tree =
   TreeNode
    (More (TreeLeaf 5,
           (More (TreeNode
                  (More (TreeLeaf 3,
                         Last (TreeLeaf 2))),
                  Last (TreeLeaf 7)))));;
```

Mutually Recursive Types - Values



Mutually Recursive Types - Values

A more conventional picture



Mutually Recursive Functions

```
# let rec fringe tree =
    match tree with
       (TreeLeaf x) -> [x]
   | (TreeNode list) -> list_fringe list
and list_fringe tree_list =
    match tree_list with
       (Last tree) -> fringe tree
   | (More (tree,list)) ->
       (fringe tree) @ (list_fringe list);;
```

val fringe : 'a tree -> 'a list = <fun>
val list_fringe : 'a treeList -> 'a list = <fun>

Mutually Recursive Functions

```
# fringe tree;;
- : int list = [5; 3; 2; 7]
```

Problem

```
# type 'a tree = TreeLeaf of ’a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size

Problem

```
# type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size

```
let rec tree_size t =
      match t with TreeLeaf _ ->
      | TreeNode ts ->
```

Problem

```
# type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size

```
let rec tree_size t =
      match t with TreeLeaf _ -> 1
      | TreeNode ts -> treeList_size  ts
```

Problem

```
# type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size and treeList_size

```
let rec tree_size t =
      match t with TreeLeaf _ -> 1
      | TreeNode ts -> treeList_size  ts
and treeList_size ts =
```

Problem

```
# type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size and treeList_size

```
let rec tree_size t =
      match t with TreeLeaf _ -> 1
      | TreeNode ts -> treeList_size  ts
and treeList_size ts =
      match ts with Last t ->
      | More t ts' ->
```

Problem

```
# type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
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- Define tree_size and treeList_size

```
let rec tree_size t =
      match t with TreeLeaf _ -> 1
      | TreeNode ts -> treeList_size  ts
and treeList_size ts =
      match ts with Last t -> tree_size t
      | More t ts’ -> tree_size t +
                      treeList_size ts’
```

Problem

```
# type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size and treeList_size

```
let rec tree_size t =
      match t with TreeLeaf _ -> 1
      | TreeNode ts -> treeList_size  ts
and treeList_size ts =
      match ts with Last t -> tree_size t
      | More t ts’ -> tree_size t +
                          treeList_size ts’
```

Nested Recursive Types

```
# type intlist =
        Nil | Cons of (int * intlist)

# type ‘a mylist =
        Nil | Cons of (‘a * ‘a mylist)
```

From the standard library: can use “type list”

```
# let x = [3] ;;
- val x : int list = [3]

# let (x : int list) = [3] ;;
- val x : int list = [3]
```

Nested Recursive Types

```
# type 'a labeled_tree =
 TreeNode of ('a * 'a labeled_tree list);;
```

```
type 'a labeled_tree =
  TreeNode of ('a * 'a labeled_tree list)
```

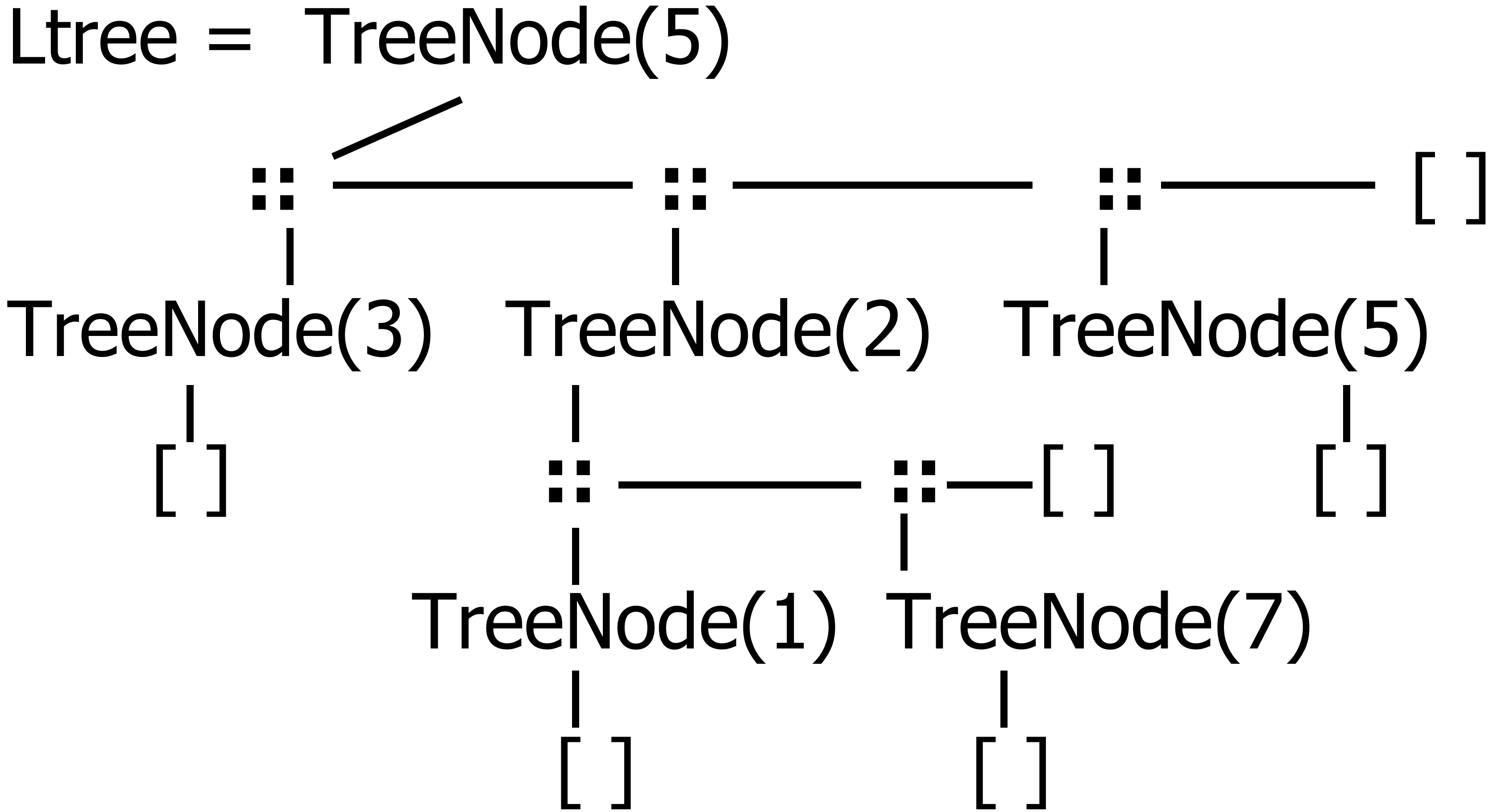
Compare:

```
# type 'a tree =
        TreeLeaf of 'a
      | TreeNode of 'a treeList
and 'a treeList =
        Last of 'a tree
      | More of ('a tree * 'a treeList);;
```

Nested Recursive Type Values

```
# let ltree =
  TreeNode(5,
    [TreeNode (3, []);
     TreeNode (2, [TreeNode (1, []);
                   TreeNode (7, [])]);
     TreeNode (5, [])]);;
```

Nested Recursive Type Values



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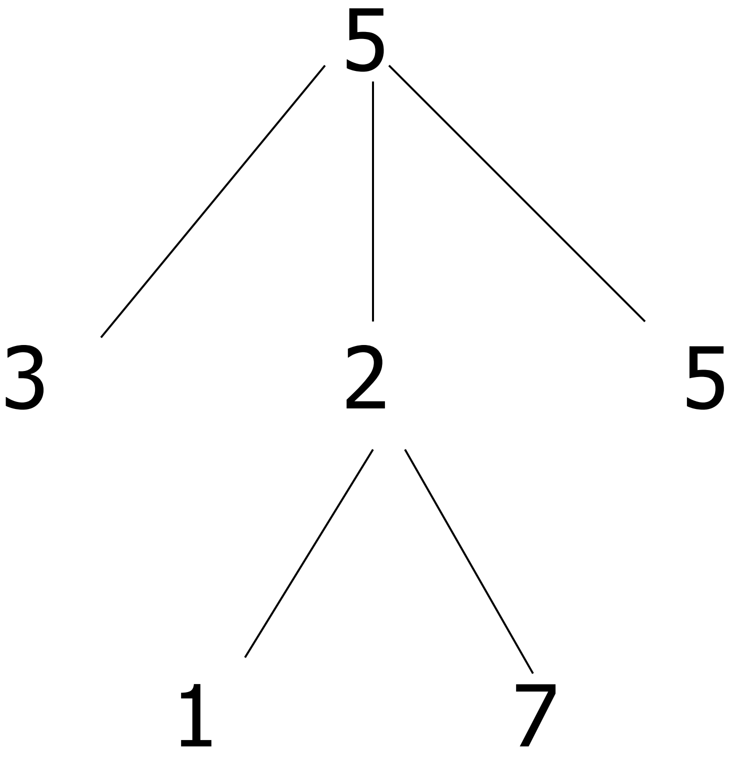

Mutually Recursive Functions

```
# let rec flatten_tree labtree =
    match labtree with
       TreeNode (x,treelist) ->
            x::flatten_tree_list treelist

  and flatten_tree_list treelist =
     match treelist with
       [] -> []
     | labtree::labtrees ->
         flatten_tree labtree
              @ (flatten_tree_list labtrees);;
```

Mutually Recursive Functions

```
val flatten_tree : 'a labeled_tree -> 'a list = <fun>
val flatten_tree_list : 'a labeled_tree list -> 'a list =
   <fun>
```

```
# flatten_tree ltree;;
-  : int list = [5; 3; 2; 1; 7; 5]
```

- **Nested recursive types lead to mutually recursive functions**

Why Data Types?

- Data types play a key role in:
  - *Data abstraction* in the design of programs
  - *Type checking* in the analysis of programs
  - *Compile-time code generation* in the translation and execution of programs
    - Data layout (how many words; which are data and which are pointers) dictated by type

Terminology

- Type: A type *t* defines a set of possible data values
  - E.g. short in C is $\{x \mid 2^{15} - 1 \geq x \geq -2^{15}\}$
  - A value in this set is said to have type *t*

- Type system: rules of a language assigning types to expressions

Types as Specifications

- Types describe properties
- Different type systems describe different properties, eg
  - Data is read-write versus read-only
  - Operation has authority to access data
  - Data came from “right” source
  - Operation might or could not raise an exception
- Common type systems focus on types describing same data layout and access methods

Sound Type System

- Type: A type *t* defines a set of possible data values
  - E.g. short in C is $\{x \mid 2^{15} - 1 \geq x \geq -2^{15}\}$
  - A value in this set is said to have type *t*
- Type system: rules of a language assigning types to expressions

- If an expression is assigned type *t*, and it evaluates to a value *v*, then *v* is in the set of values defined by *t*

Sound Type System

If an expression is assigned type *t*, and it evaluates to a value *v*, then *v* is in the set of values defined by *t*

For instance:

- `let x = true in let y = true in let z = x && y`
- `let x = 5 in let y = 6 + x`
- `let r = 2.0 in let w = 3.14 *. 2.0 *. r`

Sound Type System

- SML, OCAML, Rust, Scheme and Ada have sound type systems (as far as we know)
- Most implementations of C and C++ do not
  - But Java and Scala are also (slightly) unsound

```
class Unsound {
  static class Constrain<A, B extends A> {}
  static class Bind<A> {
    <B extends A>
    A upcast(Constrain<A,B> constrain, B b) {
      return b;
    }
  }
  static <T,U> U coerce(T t) {
    Constrain<U,? super T> constrain = null;
    Bind<U> bind = new Bind<U>();
    return bind.upcast(constrain, t);
  }
  public static void main(String[] args) {
    String zero = Unsound.<Integer,String>coerce(0);
  }
}
```

Figure 1. Unsound valid Java program compiled by javac, version 1.8.0_25

- For details, see this paper: Java and Scala's Type Systems are Unsound ∗ The Existential Crisis of Null Pointers. Amin and Tate (OOPSLA 2016)

Strongly Typed Language

- When no application of an operator to arguments can lead to a run-time type error, language is *strongly typed*
  - Eg: 1 + 2.3;;
- Depends on definition of “type error”

Strongly Typed Language

- When no application of an operator to arguments can lead to a run-time type error, language is **strongly typed**
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- Depends on definition of “type error”

Strongly Typed Language

- C++ claimed to be “strongly typed”, but
  - Union types allow creating a value at one type and using it at another
  - Type coercions may cause unexpected (undesirable) effects
  - No array bounds check. In fact, no runtime checks at all.

- SML, OCAML “strongly typed” but still must do dynamic array bounds checks, runtime type case analysis, and other checks

Static vs Dynamic Types

- ***Static type*** : type assigned to an expression at compile time
- ***Dynamic type*** : type assigned to a storage location at run time
- ***Statically typed language*** : static type assigned to every expression at compile time
- ***Dynamically typed language*** : type of an expression determined at run time

Type Checking

- When is op(arg1,…,argn) allowed?
- **Type checking** assures that operations are applied to the right number of arguments of the right types
  - Right type may mean same type as was specified, or may mean that there is a predefined implicit coercion that will be applied
- Used to resolve overloaded operations

Type Checking

- Type checking may be done ***statically*** at compile time or ***dynamically*** at run time
- Dynamically typed (aka untyped) languages (e.g., LISP, Prolog) do only dynamic type checking
- Statically typed languages can do most type checking statically

Dynamic Type Checking

- Performed at run-time before each operation is applied
- Types of variables and operations left unspecified until run-time
  - Same variable may be used at different types

Dynamic Type Checking

- Data object must contain type information
- Errors aren't detected until violating application is executed (maybe years after the code was written)

Static Type Checking

- Performed after parsing, before code generation
- Type of every variable and signature of every operator must be known at compile time

Static Type Checking

- Can eliminate need to store type information in data object if no dynamic type checking is needed
- Catches many programming errors at earliest point
- Can't check types that depend on dynamically computed values
  - Eg: array bounds

Static Type Checking

- Typically places restrictions on languages
  - Garbage collection
  - References instead of pointers
  - All variables initialized when created
  - Variable only used at one type
    - Union types allow for work-arounds, but effectively introduce dynamic type checks

Type Declarations

- *Type declarations*: explicit assignment of types to variables (signatures to functions) in the code of a program
  - Must be checked in a strongly typed language
  - Often not necessary for strong typing or even static typing (depends on the type system)

Type Inference

- *Type inference*: A program analysis to assign a type to an expression from the program context of the expression
  - Fully static type inference first introduced by Robin Miller in ML
  - Haskle, OCAML, SML all use type inference
    - Records are a problem for type inference

Format of Type Judgments

- A *type judgement* has the form

$$\Gamma \mid\!- \text{exp} : \tau$$

- Γ is a typing environment
  - Supplies the types of variables (and function names when function names are not variables)
  - Γ is a set of the form $\{ x : \sigma , \ldots \}$
  - For any x at most one σ such that $(x : \sigma \in \Gamma)$
- exp is a program expression
- τ is a type to be assigned to exp
- |- pronounced "turnstyle", or "entails" (or "satisfies" or, informally, "shows")

Inductive Proof System

- Hypotheses and Conclusion are logical formulas
- **Inference Rule:** Hypotheses imply Conclusion

$$\frac{\text{Hypothesis\_1} \qquad \text{Hypothesis\_2} \quad \ldots \quad \text{Hypothesis\_n}}{\text{Conclusion}}$$

- **Axiom:** Holds without any previous hypothesis

$$\frac{}{\text{Conclusion}}$$

- Analogy: Axiom as a base case, Theorem as an inductive case, Proof as a recursive derivation

Axioms – Constants (Monomorphic)

$$\frac{}{\Gamma \mid\!- \; n : \text{int}}$$ (assuming *n* is an integer constant)

$$\frac{}{\Gamma \mid\!- \; \text{true} : \text{bool}} \qquad \frac{}{\Gamma \mid\!- \; \text{false} : \text{bool}}$$

- These rules are true with any typing environment
- Γ, *n* are meta-variables

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- Γ, *n* are meta-variables

Axioms – Variables (Monomorphic Rule)

Notation: Let $\Gamma(x) = \sigma$ if $x : \sigma \in \Gamma$

Note: if such σ exits, its unique

Variable axiom:

$$\frac{}{\Gamma \mid\!- \; x : \sigma} \quad \text{if } \Gamma(x) = \sigma$$

- The predicate $\Gamma(x) = \sigma$ is defined such that it is false if x has different type or x is not defined.

Simple Rules – Arithmetic (Example)

Primitive Binary operators:

$$\frac{\Gamma \mid\!- e_1 : \text{int} \quad \Gamma \mid\!- e_2 : \text{int} \quad (+) : \text{int} \rightarrow \text{int} \rightarrow \text{int}}{\Gamma \mid\!- e_1 + e_2 : \text{int}}$$

Relations:

$$\frac{\Gamma \mid\!- e_1 : \text{int} \quad \Gamma \mid\!- e_2 : \text{int} \quad (=) : \text{int} \rightarrow \text{int} \rightarrow \text{bool}}{\Gamma \mid\!- e_1 = e_2 : \text{bool}}$$

Simple Rules – Arithmetic (Mono)

Primitive Binary operators ($\oplus \in \{ +, -, *, \ldots\}$):

$$\frac{\Gamma \mid\!- e_1 : \tau_1 \qquad \Gamma \mid\!- e_2 : \tau_2 \qquad (\oplus) : \tau_1 \rightarrow \tau_2 \rightarrow \tau_3}{\Gamma \mid\!- e_1 \oplus e_2 : \tau_3}$$

Special case: Relations ($\sim \in \{ <, >, =, <=, >= \}$):

$$\frac{\Gamma \mid\!- e_1 : \tau \quad \Gamma \mid\!- e_2 : \tau \quad (\sim) : \tau \rightarrow \tau \rightarrow \text{bool}}{\Gamma \mid\!- e_1 \sim e_2 : \text{bool}}$$

All τ are **type variables**

For the moment, think τ is int or bool

Example: {x:int} |- x + 2 = 3 :bool

What do we need to show first?

{x:int} |- x + 2 = 3 : bool

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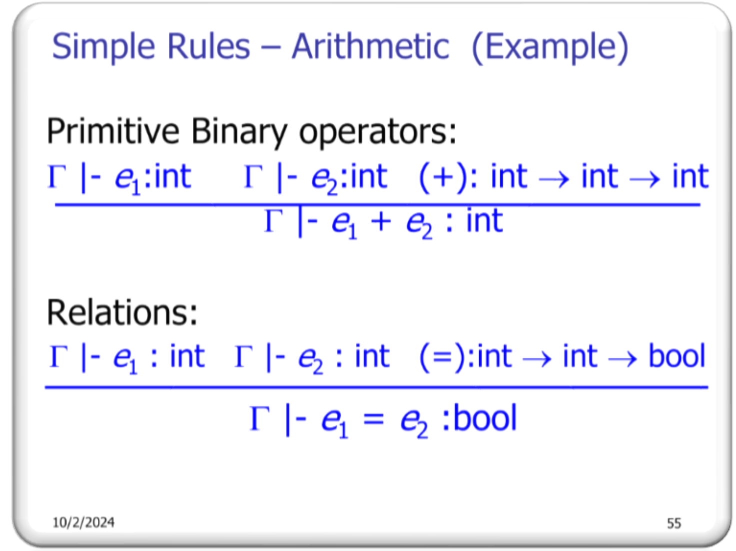

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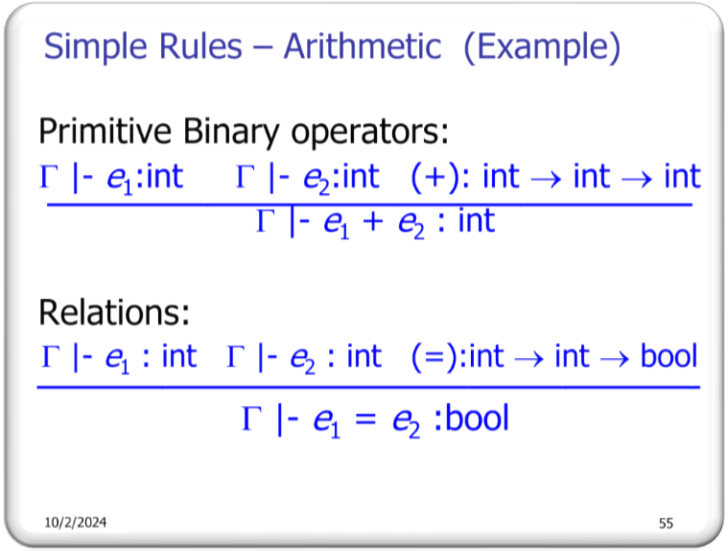
Example: {x:int} |- x + 2 = 3 :bool

What do we need for the left side?

$$\frac{\{x : int\} \mid\!- x + 2 : int \qquad\qquad \{x:int\} \mid\!- 3 :int}{\{x:int\} \mid\!- x + 2 = 3 : bool}\text{Bin}$$

Example: {x:int} |- x + 2 = 3 :bool



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$$\dfrac{\{x : \text{int}\} \mid\!- \; x + 2 : \text{int} \qquad \{x\text{:int}\} \mid\!- \; 3 \text{ :int}}{\{x\text{:int}\} \mid\!- \; x + 2 = 3 : \text{bool}} \text{Bin}$$

Example: {x:int} |- x + 2 = 3 :bool

How to finish?

$$\dfrac{\dfrac{\{x:int\} \mid\!- x:int \quad \{x:int\} \mid\!- 2:int}{\{x : int\} \mid\!- x + 2 : int}\text{Bin} \qquad \{x:int\} \mid\!- 3 :int}{\{x:int\} \mid\!- x + 2 = 3 : bool}\text{Bin}$$

Example: {x:int} |- x + 2 = 3 :bool

How to finish?

Axioms – Constants (Monomorphic)

$$\frac{}{\Gamma \ |\text{-}\ n : \text{int}}$$ (assuming *n* is an integer constant)

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$$\dfrac{\dfrac{\{x:int\} \ |\text{-}\ x:int \quad \{x:int\} \ |\text{-}\ 2:int}{\{x : int\} \ |\text{-}\ x + 2 : int}\text{Bin} \qquad \{x:int\} \ |\text{-}\ 3 :int}{\{x:int\} \ |\text{-}\ x + 2 = 3 : bool}\text{Bin}$$

Example: {x:int} |- x + 2 = 3 :bool

Almost Complete Proof
(type derivation)

$$\dfrac{\{x:int\} \vdash x:int \quad \dfrac{}{\{x:int\} \vdash 2:int}\,\text{Const}}{\{x:int\} \vdash x + 2 : int}\,\text{Bin}$$

$$\dfrac{}{\{x:int\} \vdash 3 :int}\,\text{Const}$$

$$\dfrac{\{x : int\} \vdash x + 2 : int \qquad \{x:int\} \vdash 3 :int}{\{x:int\} \vdash x + 2 = 3 : bool}\,\text{Bin}$$

Example: {x:int} |- x + 2 = 3 :bool

Almost Complete Proof
(type derivation)

$$\dfrac{\{x{:}int\} \mid\!- x{:}int \quad \dfrac{}{\{x{:}int\} \mid\!- 2{:}int}\text{Cons}}{\{x : int\} \mid\!- x + 2 : int}\text{Bin} \quad \dfrac{}{\{x{:}int\} \mid\!- 3 : int}\text{Const}$$

$$\dfrac{\{x : int\} \mid\!- x + 2 : int \qquad \{x{:}int\} \mid\!- 3 : int}{\{x{:}int\} \mid\!- x + 2 = 3 : bool}\text{Bin}$$

Axioms – Variables (Monomorphic Rule)

Notation: Let $\Gamma(x) = \sigma$ if $x : \sigma \in \Gamma$

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Variable axiom:

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Example: {x:int} |- x + 2 = 3 :bool

Complete Proof (type derivation)

$$\dfrac{\dfrac{\dfrac{}{\{x:int\} \vdash x:int}\text{Var} \quad \dfrac{}{\{x:int\} \vdash 2:int}\text{Const}}{\{x : int\} \vdash x + 2 : int}\text{Bin} \quad \dfrac{}{\{x:int\} \vdash 3 :int}\text{Const}}{\{x:int\} \vdash x + 2 = 3 : bool}\text{Bin}$$

Simple Rules - Booleans

Connectives

$$\frac{\Gamma \text{ |- } e_1 : \text{bool} \qquad \Gamma \text{ |- } e_2 : \text{bool}}{\Gamma \text{ |- } e_1 \text{ \&\& } e_2 : \text{bool}}$$

$$\frac{\Gamma \text{ |- } e_1 : \text{bool} \qquad \Gamma \text{ |- } e_2 : \text{bool}}{\Gamma \text{ |- } e_1 \text{ || } e_2 : \text{bool}}$$

Conditionals?

- If_then_else rule:

$$\frac{?}{\Gamma \mid\!- (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

- If the conditional expression has type τ , then what should the types of subexpressions be?

Conditionals?

- If_then_else rule:

$$\frac{\Gamma \mid\!-\; e_1 : ? \quad \Gamma \mid\!-\; e_2 : ? \quad \Gamma \mid\!-\; e_3 : ?}{\Gamma \mid\!-\; (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

Conditionals?

- If_then_else rule:

$$\frac{\Gamma \mid\!- \ e_1 : \text{bool} \quad \Gamma \mid\!- \ e_2 : \tau \quad \Gamma \mid\!- \ e_3 : \tau}{\Gamma \mid\!- \ (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

Type Variables in Rules

- If_then_else rule:

$$\frac{\Gamma \mid\!- \; e_1 : \text{bool} \quad \Gamma \mid\!- \; e_2 : \tau \quad \Gamma \mid\!- \; e_3 : \tau}{\Gamma \mid\!- \; (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

- τ is a type variable (meta-variable)
- Can take any type at all
- All instances in a rule application must get same type
- Then branch, else branch and if_then_else must all have same type

Example derivation: if-then-else-

- Γ = {x:int, int_of_float:float -> int, y:float}

Type Variables in Rules

- If_then_else rule:

$$\frac{\Gamma \text{ |- } e_1 : \text{bool} \quad \Gamma \text{ |- } e_2 : \tau \quad \Gamma \text{ |- } e_3 : \tau}{\Gamma \text{ |- } (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

$$\frac{}{\Gamma \text{ |- if } x > 3 \text{ then } x + 2 \text{ else int\_of\_float } y : \text{int}}$$

Example derivation: if-then-else-

- Γ = {x:int, int_of_float:float -> int, y:float}

$$\frac{\Gamma \mid\!- \; x > 3 : \text{bool} \qquad \Gamma \mid\!- \; x+2 : \text{int} \qquad \Gamma \mid\!- \; \text{int\_of\_float } y : \text{int}}{\Gamma \mid\!- \; \text{if } x > 3 \text{ then } x + 2 \text{ else int\_of\_float } y : \text{int}}$$

Function Application?

- Application rule:

$$\frac{?}{\Gamma \;|\!\text{-}\; (e_1\ e_2) : \tau_2}$$

- If the function application has type τ_2 , then what should the types of subexpressions be?

Function Application?

- Application rule:

$$\frac{\Gamma \mid\!- \; e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \mid\!- \; e_2 : \tau_1}{\Gamma \mid\!- \; (e_1 \; e_2) : \tau_2}$$

Function Application?

- Application rule:

$$\frac{\Gamma \mid\!- \; e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \mid\!- \; e_2 : \tau_1}{\Gamma \mid\!- \; (e_1 \; e_2) : \tau_2}$$

Function Application

- Application rule:

$$\frac{\Gamma \mid\!- \; e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \mid\!- \; e_2 : \tau_1}{\Gamma \mid\!- \; (e_1 \; e_2) : \tau_2}$$

- If you have a function expression e_1 of type $\tau_1 \rightarrow \tau_2$ applied to an argument e_2 of type τ_1, the resulting expression $e_1 e_2$ has type τ_2

Example: Application

Function Application

- Application rule:

$$\frac{\Gamma \mid\!- \ e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \mid\!- \ e_2 : \tau_1}{\Gamma \mid\!- \ (e_1 \ e_2) : \tau_2}$$

- Γ = {x: int, int_of_float: float -> int, y: float}

$$\frac{\Gamma \mid\!- \text{ int\_of\_float : float -> int} \qquad \Gamma \mid\!- \text{ y : float}}{\Gamma \mid\!- \text{ int\_of\_float y : int}}$$

Example: Application

Function Application

- Application rule:

$$\frac{\Gamma \mid\!- \; e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \mid\!- \; e_2 : \tau_1}{\Gamma \mid\!- \; (e_1\ e_2) : \tau_2}$$

- Γ = {x:int, int_of_float:float -> int, y:float}

$$\frac{\Gamma \mid\!- \text{(fun z -> z > 3) : int -> bool} \qquad \Gamma \mid\!- \text{x : int}}{\Gamma \mid\!- \text{(fun z -> z > 3) x : bool}}$$

Function Abstraction?

- Fun rule:

$$\frac{?}{\Gamma \mid\!\!- \text{fun } x \text{ -> } e : \tau_1 \rightarrow \tau_2}$$

Function Abstraction?

- Fun rule:

(1) We add x to the typing environment
(2) We check that e has the proper type

$$\frac{\text{(1) We add x to the typing environment} \quad \text{(2) We check that e has the proper type}}{\Gamma \mid\!- \text{fun } x \text{ -> } e : \tau_1 \rightarrow \tau_2}$$

Function Abstraction?

- Fun rule:

$$\frac{\text{(1) We add x to the environment with type } \tau_1 \quad \text{(2) We check that e has the type } \tau_2}{\Gamma \text{ |- fun } x \text{ -> } e : \tau_1 \rightarrow \tau_2}$$

Fun Rule

- Rules describe types, but also how the environment Γ may change
- Can only do what rule allows!
- fun rule:

$$\frac{\{x : \tau_1\} + \Gamma \mid\!- \; e : \tau_2}{\Gamma \mid\!- \text{fun } x \text{ -> } e : \tau_1 \rightarrow \tau_2}$$

Fun Examples

$$\frac{\{y : int\} + \Gamma \text{ |- } y + 3 : int}{\Gamma \text{ |- fun } y \text{ -> } y + 3 : int \rightarrow int}$$

$$\frac{\{f : int \rightarrow bool\} + \Gamma \text{ |- } f\ \ 2 :: [true] : bool\ list}{\Gamma \text{ |- (fun } f \text{ -> } (f\ \ 2) :: [true]) : (int \rightarrow bool) \rightarrow bool\ list}$$

How about let ?

- Let rule

$$\frac{?}{\Gamma \mid\!- (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

- Recall: how was let … in … represented with just function abstraction and application?

How about let ?

- Let rule

$$\frac{?}{\Gamma \mid\!- (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

- let x = e_1 in e_2 <====>
- (fun x -> e2) e1

(Monomorphic) Let and Let Rec

- let rule:

$$\frac{\Gamma \mid\!- \; e_1 : \tau_1 \qquad \{x : \tau_1\} + \Gamma \mid\!- \; e_2 : \tau_2}{\Gamma \mid\!- \; (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

(Monomorphic) Let and Let Rec

- let rule:

$$\frac{\Gamma \mid\!- \; e_1 : \tau_1 \qquad \{x : \tau_1\} + \Gamma \mid\!- \; e_2 : \tau_2}{\Gamma \mid\!- \; (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

- let rec rule:

$$\frac{\{f : \tau_1\} + \Gamma \mid\!- \; e_1 : \tau_1 \quad \{f : \tau_1\} + \Gamma \mid\!- \; e_2 : \tau_2}{\Gamma \mid\!- \; (\text{let rec } f = e_1 \text{ in } e_2) : \tau_2}$$

(Monomorphic) Let and Let Rec

- let rule:

$$\frac{\Gamma \mid\!\!-\ e_1 : \tau_1 \qquad \{x : \tau_1\} + \Gamma \mid\!\!-\ e_2 : \tau_2}{\Gamma \mid\!\!-\ (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

- let rec rule:

$$\frac{\{f : \tau_1\} + \Gamma \mid\!\!-\ e_1 : \tau_1 \quad \{f : \tau_1\} + \Gamma \mid\!\!-\ e_2 : \tau_2}{\Gamma \mid\!\!-\ (\text{let rec } f = e_1 \text{ in } e_2) : \tau_2}$$

Curry - Howard Isomorphism

- Type Systems are logics; logics are type systems
- Types are propositions; propositions are types
- Terms are proofs; proofs are terms

- Function space arrow corresponds to implication; application corresponds to modus ponens

Curry - Howard Isomorphism

- Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B}$$

- Application

$$\frac{\Gamma \mid\!- \; e_1 : \alpha \rightarrow \beta \quad \Gamma \mid\!- \; e_2 : \alpha}{\Gamma \mid\!- \; (e_1 \; e_2) : \beta}$$