Programming Languages and Compilers (CS 421)

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https://courses.engr.illinois.edu/cs421/fa2024/CS421C

Based on slides by Elsa Gunter, which are based in part on previous slides by Mattox Beckman and updated by Vikram Adve and Gul Agha

CPS Transformation

- **Step 1:** Add continuation argument to any function definition:
  - `let f arg = e` $\Rightarrow$ `let f arg k = e`
  - Idea: Every function takes an extra parameter saying where the result goes

- **Step 2:** A simple expression in tail position should be passed to a continuation instead of returned:
  - `return a` $\Rightarrow$ `k a`
  - Assuming a is a constant or variable.
  - “Simple” = “No available function calls.”

CPS Transformation

- **Step 3:** Pass the current continuation to every function call in tail position
  - `return f arg` ⇒ `f arg k`
  - The function "isn't going to return," so we need to tell it where to put the result.

CPS Transformation

- **Step 4:** Each function call not in tail position needs to be converted to take a new continuation (containing the old continuation as appropriate)
  - `return op (f arg)` ⇒ `f arg (fun r -> k(op r))`
    - op represents a primitive operation
  - `return  f(g arg)` ⇒ `g arg (fun r-> f r k)`

Example

Step 1: Add continuation argument to any function definition
Step 2: A simple expression in tail position should be passed to a continuation instead of returned
Step 3: Pass the current continuation to every function call in tail position
Step 4: Each function call not in tail position needs to be converted to take a new continuation (containing the old continuation as appropriate)

Before:

```
let rec add_list lst =

match lst with
  [ ] -> 0
| 0 :: xs -> add_list xs
| x :: xs -> (+) x
   (add_list xs);;
```

After:

```
let rec add_listk lst k =
                                  (* rule 1 *)
match lst with
| [ ] -> k 0                      (* rule 2 *)
| 0 :: xs -> add_listk xs k
                                  (* rule 3 *)
| x :: xs -> add_listk xs
          (fun r -> k ((+) x r));;
                                  (* rule 4 *)
```

Same as:

```
fun r -> addk x r k
```

Example

Before:

```
let rec mem (y,lst) =
match lst with
  [ ] -> false
| x :: xs ->
  if (x = y)
   then true
   else mem(y,xs);;
```

After:

```
let rec memk (y,lst) k =
                    (* rule 1 *)
```

Example

Before:

```
let rec mem (y,lst) =
match lst with
  [ ] -> false
| x :: xs ->
  if (x = y)
   then true
   else mem(y,xs);;
```

After:

```
let rec memk (y,lst) k =
                    (* rule 1 *)

                k false (* rule 2 *)



           k true (* rule 2 *)
```

Example

Before:

```
let rec mem (y,lst) =
match lst with
  [ ] -> false
| x :: xs ->
  if (x = y)
   then true
   else mem(y,xs);;
```

After:

```
let rec memk (y,lst) k =
                  (* rule 1 *)

          k false (* rule 2 *)




     k true (* rule 2 *)
        memk (y, xs) k (* rule 3
```

Example

Before:

```
let rec mem (y,lst) =
match lst with
  [ ] -> false
| x :: xs ->
  if (x = y)
   then true
   else mem(y,xs);;
```

After:

```
let rec memk (y,lst) k =
                (* rule 1 *)

         k false (* rule 2 *)

 eqk (x, y)
  (fun b ->   b (* rule 4 *)
     k true (* rule 2 *)
        memk (y, xs) (* rule 3 *
```

Example

Before:

```
let rec mem (y,lst) =
match lst with
  [ ] -> false
| x :: xs ->
  if (x = y)
   then true
   else mem(y,xs);;
```

After:

```
let rec memk (y,lst) k =
                               (* rule 1 *)

                    k false (* rule 2 *)


 eqk (x, y)
  (fun b ->if b (* rule 4 *)
then k true (* rule 2 *)
   else memk (y, xs) (* rule 3 *
```

Example

Before:

```
let rec mem (y,lst) =
match lst with
  [ ] -> false
| x :: xs ->
  if (x = y)
   then true
   else mem(y,xs);;
```

After:

```
let rec memk (y,lst) k =
                       (* rule 1 *)
match lst with
  [ ] -> k false (* rule 2 *)
| x :: xs ->
 eqk (x, y)
  (fun b ->if b (* rule 4 *)
then k true (* rule 2 *)
   else memk (y, xs) k (* rule 3
```

Other Uses for Continuations

- CPS designed to **preserve evaluation order**
- **Continuations** used to **express** order of evaluation

- Can also be used to **change** order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

Exceptions - Example

```
# exception Zero;;
exception Zero

# let rec list_mult_aux list =
    match list with
      [ ] -> 1
    | x :: xs ->
           if x = 0 then raise Zero
                    else x * list_mult_aux xs;;
val list_mult_aux : int list -> int = <fun>
```

Exceptions - Example

```
# let list_mult list =
    try list_mult_aux list with Zero -> 0;;
val list_mult : int list -> int = <fun>

# list_mult [3;4;2];;
- : int = 24

# list_mult [7;4;0];;
- : int = 0

# list_mult_aux [7;4;0];;
Exception: Zero.
```

Exceptions

- When an exception is raised
  - The current computation is aborted
  - Control is “thrown” back up the call stack until a matching handler is found
  - All the intermediate calls waiting for a return values are thrown away

Implementing Exceptions

```
# let multkp (m, n) k =
    let r = m * n in
      ( print_string "product result: ";
        print_int r; print_string "\n";
        k r);;
val multkp : int ( int -> (int -> 'a) -> 'a =
  <fun>
```

Implementing Exceptions

```
# let rec list_multk_aux list k kexcp =
    match list with
       [ ] -> k 1
     | x :: xs -> if x = 0 then kexcp 0
                  else
                    list_multk_aux xs
                        (fun r -> multkp (x, r) k)
                        kexcp;;

# let rec list_multk list k =
       list_multk_aux list k
                 (fun x -> print_string "nil\n");;
```

Implementing Exceptions

```
# list_multk [3;4;2] report;;
product result: 2
product result: 8
product result: 24
24
- : unit = ()

# list_multk [7;4;0] report;;
nil
- : unit = ()
```

Advanced: Using CPS as Compiler Intermediate Representation

Ocaml compiler (latest version) uses CPS:

- Blog: https://discuss.ocaml.org/t/blog-the-flambda2-snippets-by-ocamlpro/14331
- Tutorial: https://www.youtube.com/watch?v=eI5GBpT2Brs

Various discussions in research literature:

- With? https://www.microsoft.com/en-us/research/wp-content/uploads/2007/10/compilingwithcontinuationscontinued.pdf
- Without? https://pauldownen.com/publications/pldi17.pdf
- Whatever? https://dl.acm.org/doi/10.1145/3341643

Intermediate representations CPS for functional vs SSA for imperative

- https://dl.acm.org/doi/10.1145/202530.202532

Data type in Ocaml: lists

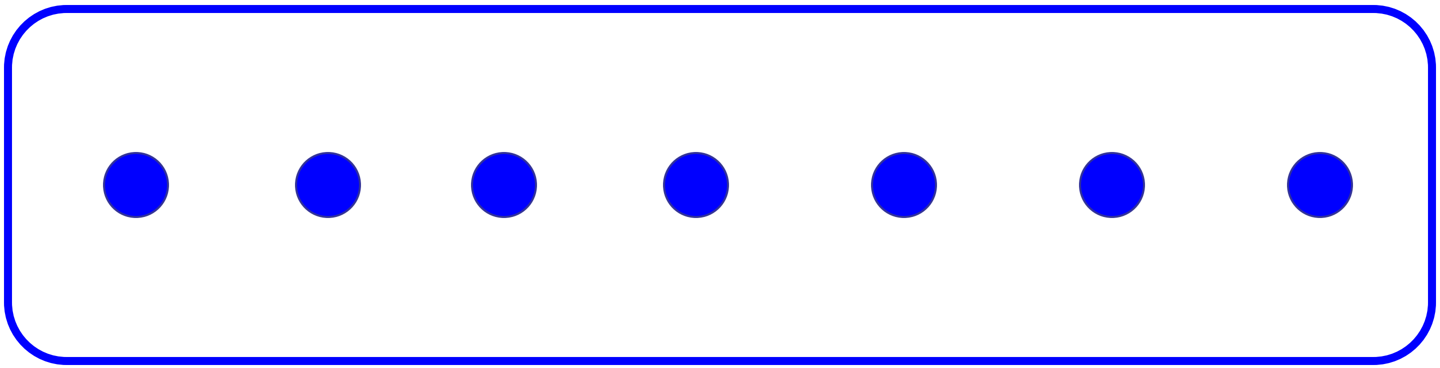
- Frequently used lists in recursive program
- Matched over two structural cases
  - [ ] - the empty list
  - (x :: xs) a non-empty list
- Covers all possible lists
- type ‘a list = [ ] | (::) of ‘a * ‘a list
  - Not quite legitimate declaration because of special syntax

Variants - Syntax (slightly simplified)

- type *name* = C_1 [of ty_1] | . . . | C_n [of ty_n]
- Introduce a type called *name*
- (fun x -> C_i x) : ty_1 -> *name*
- C_i is called a *constructor*; if the optional type argument is omitted, it is called a *constant*
- Constructors are the basis of almost all pattern matching

Enumeration Types as Variants

An enumeration type is a collection of distinct values



In C and Ocaml they have an order structure; order by order of input

Enumeration Types as Variants

```
# type weekday = Monday | Tuesday | Wednesday
   | Thursday | Friday | Saturday | Sunday;;
type weekday =
    Monday
  | Tuesday
  | Wednesday
  | Thursday
  | Friday
  | Saturday
  | Sunday
```

Functions over Enumerations

```
# let day_after day = match day with
    Monday -> Tuesday
  | Tuesday -> Wednesday
  | Wednesday -> Thursday
  | Thursday -> Friday
  | Friday -> Saturday
  | Saturday -> Sunday
  | Sunday -> Monday;;
 val day_after : weekday -> weekday = <fun>
```

Functions over Enumerations

```
# type weekday = Monday | Tuesday |
                 Wednesday | Thursday |
                 Friday | Saturday | Sunday;;
```

Write a function days_later n day that computes a day which is n days away from the day. Note that n can be greater than 7 (more than one week) and also negative (meaning a day before

```
# let rec days_later n day =
    match n with
      0 -> day
    | _ -> if n > 0
           then day_after (days_later (n - 1) day)
          else days_later (n + 7) day;;
val days_later : int -> weekday -> weekday=<fun>
```

Functions over Enumerations

```
# days_later 2 Tuesday;;
- : weekday = Thursday

# days_later (-1) Wednesday;;
- : weekday = Tuesday

# days_later (-4) Monday;;
- : weekday = Thursday
```

Problem:

```
# type weekday = Monday | Tuesday | Wednesday
    | Thursday | Friday | Saturday | Sunday;;
```

- Write function is_weekend : weekday -> bool

Problem:

```
# type weekday = Monday | Tuesday | Wednesday
   | Thursday | Friday | Saturday | Sunday;;
```

- Write function is_weekend : weekday -> bool

```
let is_weekend day =
     match day with
        Saturday -> true
      | Sunday -> true
      | _ -> false
```

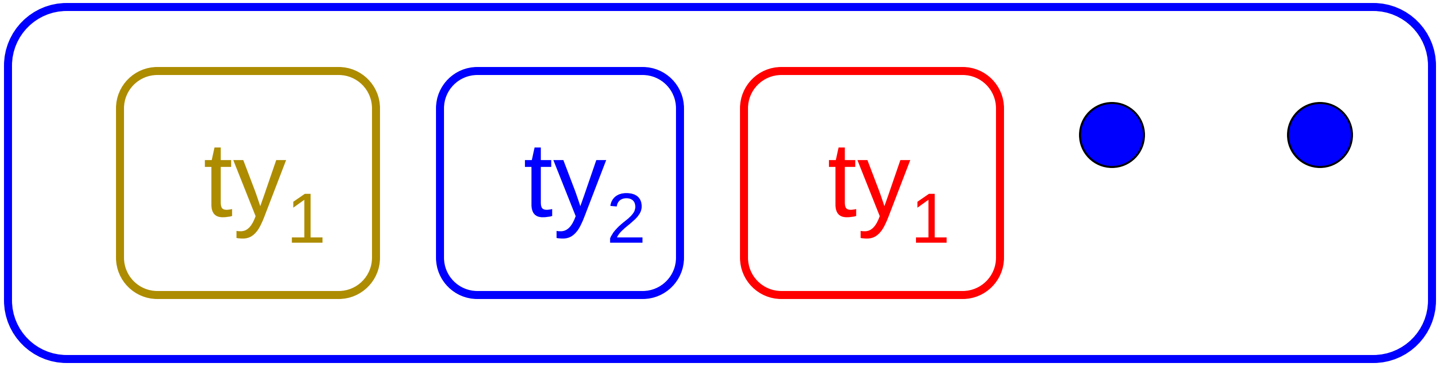
Example Enumeration Types

```
# type bin_op = IntPlusOp |  IntMinusOp
          |  EqOp | CommaOp | ConsOp

# type mon_op = HdOp | TlOp | FstOp
          | SndOp
```

Disjoint Union Types

- **Disjoint union of types**, with some possibly occurring more than once



- We can also add in some new singleton elements

Disjoint Union Types

```
# type id = DriversLicense of int
  | SocialSecurity of int | Name of string;;
type id = DriversLicense of int |
  SocialSecurity of int | Name of string

# let check_id id =
    match id with
      DriversLicense num ->
       not (List.mem num [13570; 99999])
    | SocialSecurity num -> num < 900000000
    | Name str -> not (str = "John Doe");;
 val check_id : id -> bool = <fun>
```

Problem

- Create a type to represent the currencies for US, UK, Europe and Japan
  - Hint: Dollar, Pound, Euro, Yen

Problem

- Create a type to represent the currencies for US, UK, Europe and Japan

```
type currency =
    Dollar of int
  | Pound of int
  | Euro of int
  | Yen of int
```

Example Disjoint Union Type

```
# type const =
        BoolConst of bool
      | IntConst of int
      | FloatConst of float
      | StringConst of string
      | NilConst
      | UnitConst
```

Example Disjoint Union Type

```
# type const = BoolConst of bool
    | IntConst of int | FloatConst of float
    | StringConst of string  | NilConst
    | UnitConst
```

- How to represent 7 as a const?
- Answer: IntConst 7

Polymorphism in Variants

- The type 'a option gives us something to represent non-existence or failure

```
# type 'a option = Some of 'a | None;;
type 'a option = Some of 'a | None
```

- Used to encode partial functions
- Often can replace the raising of an exception

Functions producing option

```
# type 'a option =
      Some of 'a
    | None;;
```

```
# let rec first p list =
    match list with [ ] -> None
    | (x::xs) -> if p x then Some x else first p xs;;
val first : ('a -> bool) -> 'a list -> 'a option =
  <fun>

# first (fun x -> x > 3) [1;3;4;2;5];;
- : int option = Some 4

# first (fun x -> x > 5) [1;3;4;2;5];;
- : int option = None
```

Functions over option

```
# type 'a option =
      Some of 'a
    | None;;
```

```
# let result_ok r =
   match r with None -> false
   | Some _ -> true;;
val result_ok : 'a option -> bool = <fun>

# result_ok (first (fun x -> x > 3) [1;3;4;2;5]);;
- : bool = true
# result_ok (first (fun x -> x > 5) [1;3;4;2;5]);;
- : bool = false
```

Problem

```
# type 'a option =
     Some of 'a
   | None;;
```

- Write a hd and tl on lists that doesn’t raise an exception and works at all types of lists.

Problem

```
# type 'a option =
      Some of 'a
    | None;;
```

- Write a hd and tl on lists that doesn’t raise an exception and works at all types of lists.

- ```
  let hd list =
        match list with
           [] -> None
         | (x::xs) -> Some x
  ```
- ```
  let tl list =
        match list with
           [] -> None
         | (x::xs) -> Some xs
  ```

Mapping over Variants

```
# let optionMap f opt =
    match opt with
      None -> None
    | Some x -> Some (f x);;
val optionMap : ('a -> 'b) -> 'a option -> 'b
  option = <fun>

# optionMap
  (fun x -> x - 2)
  (first (fun x -> x > 3) [1;3;4;2;5]);;
- : int option = Some 2
```

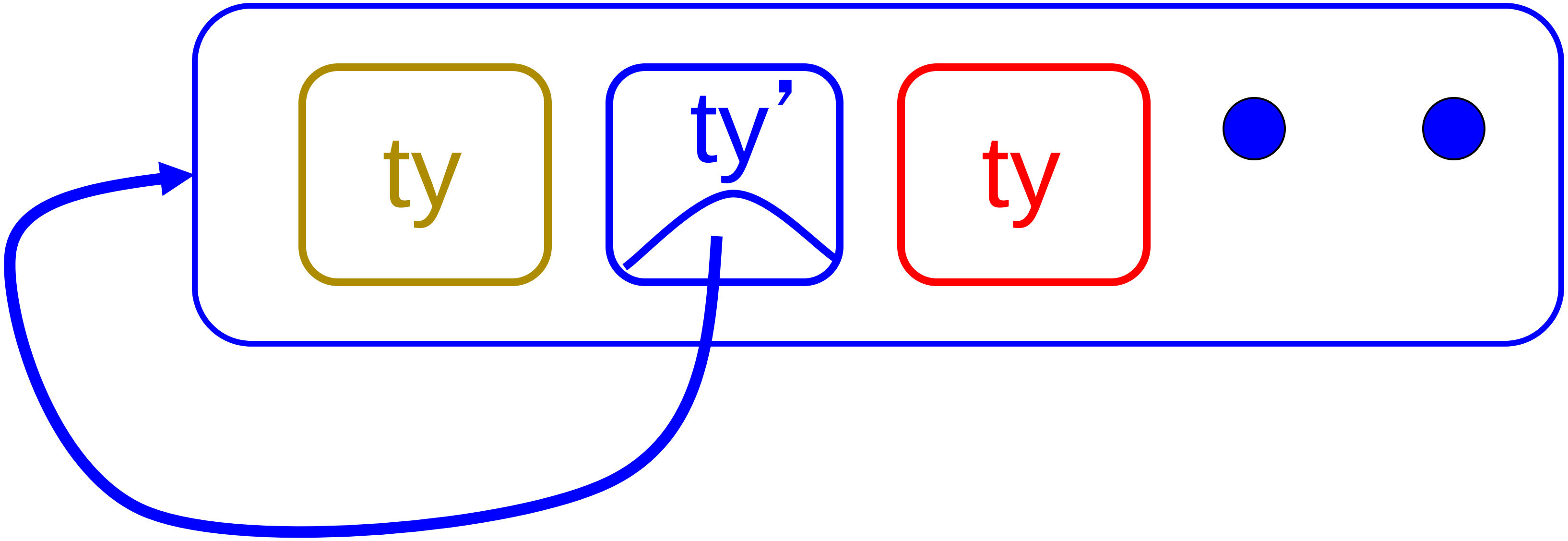
Folding over Variants

```
# let optionFold someFun noneVal opt =
    match opt with
      None -> noneVal
    | Some x -> someFun x;;
val optionFold : ('a -> 'b) -> 'b -> 'a option
  -> 'b = <fun>
```

```
# let optionMap f opt =
   optionFold (fun x -> Some (f x)) None opt;;
val optionMap : ('a -> 'b) -> 'a option -> 'b
  option = <fun>
```

Recursive Types

- The type being defined may be a component of itself



Recursive Data Types

```
# type int_Bin_Tree =
     Leaf of int
   | Node of (int_Bin_Tree * int_Bin_Tree);;
```

```
type int_Bin_Tree = Leaf of int | Node of
  (int_Bin_Tree * int_Bin_Tree)
```

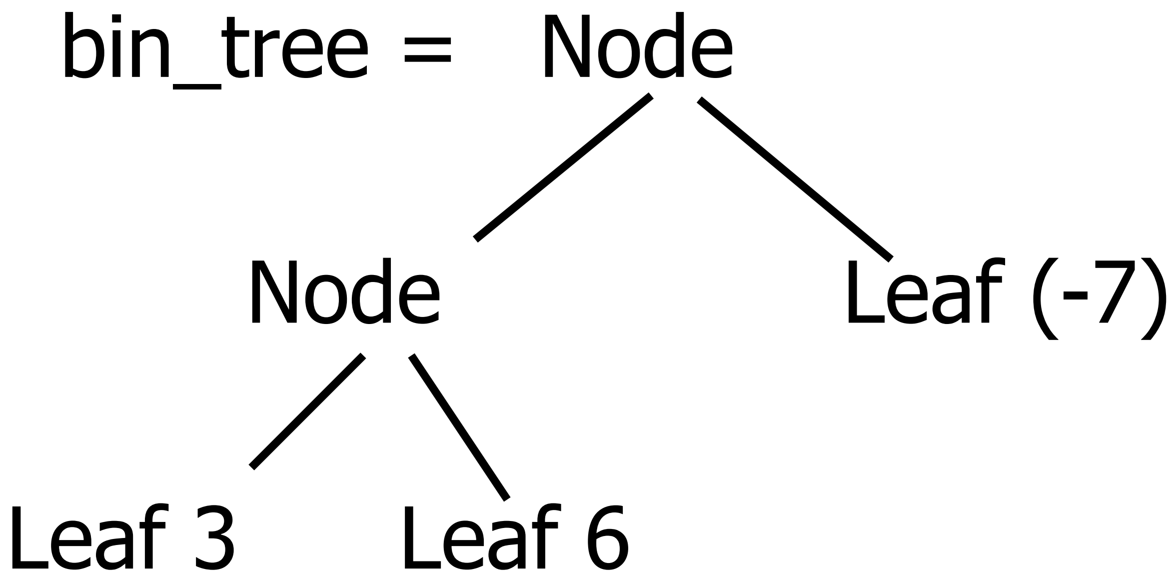
Recursive Data Type Values

```
# let bin_tree =
    Node(Node(Leaf 3, Leaf 6),Leaf (-7));;

val bin_tree : int_Bin_Tree = Node (Node
  (Leaf 3, Leaf 6), Leaf (-7))
```

Recursive Data Type Values

```
# let bin_tree =
    Node(Node(Leaf 3, Leaf 6),Leaf (-7));;
```



Recursive Functions

```
# let rec first_leaf_value tree =
    match tree with
      (Leaf n) -> n
     | Node (left_tree, right_tree) ->
            first_leaf_value left_tree;;

# let left = first_leaf_value bin_tree;;
val left : int = 3
```

Recursive Data Types

```
# type exp =
      VarExp of string
    | ConstExp of const
    | MonOpAppExp of mon_op * exp
    | BinOpAppExp of bin_op * exp * exp
    | IfExp of exp* exp * exp
    | AppExp of exp * exp
    | FunExp of string * exp
```

Recursive Data Types

```
# type bin_op = IntPlusOp |  IntMinusOp
          |  EqOp | CommaOp | ConsOp | …
# type const = BoolConst of bool | IntConst of int
| …
# type exp = VarExp of string | ConstExp of const
   | BinOpAppExp of bin_op * exp * exp  | …
```

- How to represent 6 as an exp?

Recursive Data Types

```
# type bin_op = IntPlusOp |  IntMinusOp
          |  EqOp | CommaOp | ConsOp | …
# type const = BoolConst of bool | IntConst of int
| …
# type exp = VarExp of string | ConstExp of const
   | BinOpAppExp of bin_op * exp * exp  | …
```

- How to represent 6 as an exp?
- Answer: ConstExp (IntConst 6)

Recursive Data Types

```
# type bin_op = IntPlusOp |  IntMinusOp
            |  EqOp | CommaOp | ConsOp | …
# type const = BoolConst of bool | IntConst of int
| …
# type exp = VarExp of string | ConstExp of const
   | BinOpAppExp of bin_op * exp * exp  | …
```

- How to represent (6, 3) as an exp?

Recursive Data Types

```
# type bin_op = IntPlusOp |  IntMinusOp
          |  EqOp | CommaOp | ConsOp | …
# type const = BoolConst of bool | IntConst of int
| …
# type exp = VarExp of string | ConstExp of const
   | BinOpAppExp of bin_op * exp * exp  | …
```

- How to represent (6, 3) as an exp?
- BinOpAppExp (CommaOp, ConstExp (IntConst 6),
  ConstExp (IntConst 3))

Recursive Data Types

```
# type bin_op = IntPlusOp |  IntMinusOp
          |  EqOp | CommaOp | ConsOp | …
# type const = BoolConst of bool | IntConst of int
| …
# type exp = VarExp of string | ConstExp of const
   | BinOpAppExp of bin_op * exp * exp  | …
```

- How to represent [(6, 3)] as an exp?
- BinOpAppExp (ConsOp, BinOpAppExp (CommaOp, ConstExp (IntConst 6), ConstExp (IntConst 3)), ConstExp NilConst))));;

Problem

```
type int_Bin_Tree =Leaf of int
| Node of (int_Bin_Tree * int_Bin_Tree);;
```

- Write sum_tree : int_Bin_Tree -> int
- Adds all ints in tree

```
let rec sum_tree t =
```

Solution

```
type int_Bin_Tree =Leaf of int
| Node of (int_Bin_Tree * int_Bin_Tree);;
```

- Write sum_tree : int_Bin_Tree -> int
- Adds all ints in tree

```
let rec sum_tree t =
  match t with
    Leaf n -> n
  | Node(t1,t2) -> sum_tree t1 + sum_tree t2
```

Recursion over Recursive Data Types

```
# type exp = VarExp of string
    | ConstExp of const
    | BinOpAppExp of bin_op * exp * exp
    | FunExp of string * exp
    | AppExp of exp * exp
```

- How to count the number of variables in an exp?

Recursion over Recursive Data Types

```
# type exp = VarExp of string | ConstExp of const
   | BinOpAppExp of bin_op * exp * exp
   | FunExp of string * exp | AppExp of exp * exp
```

- How to count the number of variables in an exp?

```
# let rec varCnt exp =
   match exp with
      VarExp x ->
    | ConstExp c ->
    | BinOpAppExp (b, e1, e2) ->
    | FunExp (x,e) ->
    | AppExp (e1, e2) ->
```

Recursion over Recursive Data Types

```
# type exp = VarExp of string | ConstExp of const
    | BinOpAppExp of bin_op * exp * exp
    | FunExp of string * exp | AppExp of exp * exp
```

- How to count the number of variables in an exp?

```
# let rec varCnt exp =
   match exp with
      VarExp x -> 1
    | ConstExp c -> 0
    | BinOpAppExp (b, e1, e2) -> varCnt e1 +varCnt e2
    | FunExp (x,e) -> 1 + varCnt e
    | AppExp (e1, e2) -> varCnt e1 + varCnt e2
```

Mapping over Recursive Types

```
# let rec ibtreeMap f tree =
    match tree with
      (Leaf n) ->
    | Node (left_tree, right_tree) ->
```

Mapping over Recursive Types

```
# let rec ibtreeMap f tree =
    match tree with
      (Leaf n) -> Leaf (f n)
    | Node (left_tree, right_tree) ->
           Node (ibtreeMap f left_tree,
                 IbtreeMap f right_tree);;
```

val ibtreeMap : (int -> int) -> int_Bin_Tree ->
 int_Bin_Tree = <fun>

Mapping over Recursive Types

```
# let bin_tree =
 Node(Node(Leaf 3, Leaf 6),Leaf (-7));;

# ibtreeMap ((+) 2) bin_tree;;

- : int_Bin_Tree = Node (Node (Leaf 5, Leaf
  8), Leaf (-5))
```

Summing up Elements of a Tree

```
# let rec tree_sum_0 tree =
    match tree with
      Leaf n ->

    | Node (left_tree, right_tree) ->
```

Folding over Recursive Types

```
# let rec ibtreeFoldRight leafFun nodeFun tree =
    match tree with
      Leaf n ->
    | Node (left_tree, right_tree) ->
```

val ibtreeFoldRight : (int -> 'a) -> ('a -> 'a -> 'a) -> int_Bin_Tree -> 'a = <fun>

Folding over Recursive Types

```
# let rec ibtreeFoldRight leafFun nodeFun tree =
   match tree with
     Leaf n -> leafFun n
   | Node (left_tree, right_tree) ->
     nodeFun
     (ibtreeFoldRight leafFun nodeFun left_tree)
     (ibtreeFoldRight leafFun nodeFun right_tree);;
```

val ibtreeFoldRight : (int -> 'a) -> ('a -> 'a -> 'a) -> int_Bin_Tree -> 'a = <fun>

Folding over Recursive Types

```
# let tree_sum =
    ibtreeFoldRight (fun x -> x) (+);;
val tree_sum : int_Bin_Tree -> int = <fun>

# tree_sum bin_tree;;
- : int = 2
```

Mutually Recursive Types

```
# type 'a tree =
        TreeLeaf of 'a
      | TreeNode of 'a treeList
and
    'a treeList =
        Last of 'a tree
      | More of ('a tree * 'a treeList);;
```

type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList)

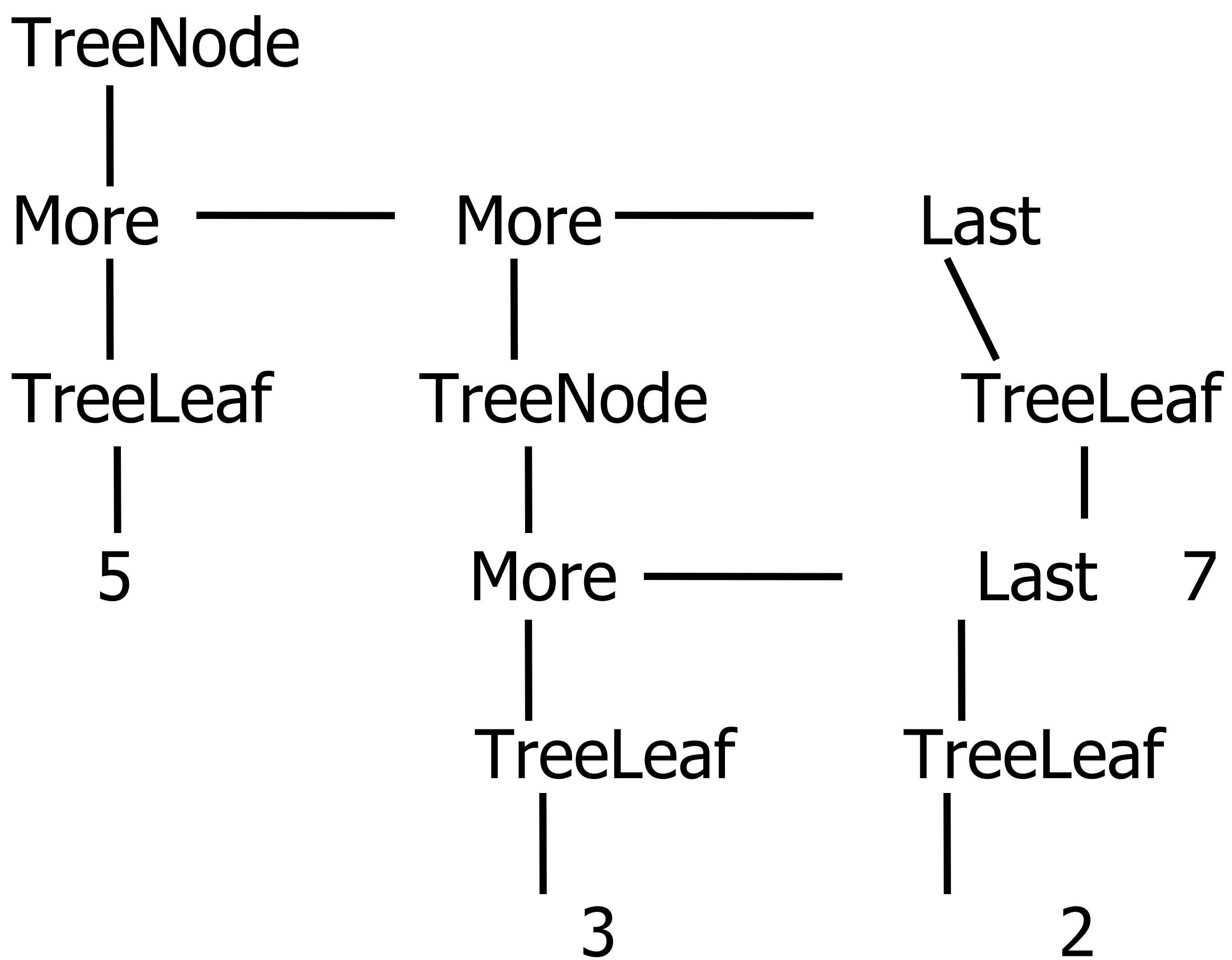
Mutually Recursive Types - Values

```
# let tree =
   TreeNode
    (More (TreeLeaf 5,
           (More (TreeNode
                  (More (TreeLeaf 3,
                         Last (TreeLeaf 2))),
                  Last (TreeLeaf 7)))));;
```

Mutually Recursive Types - Values

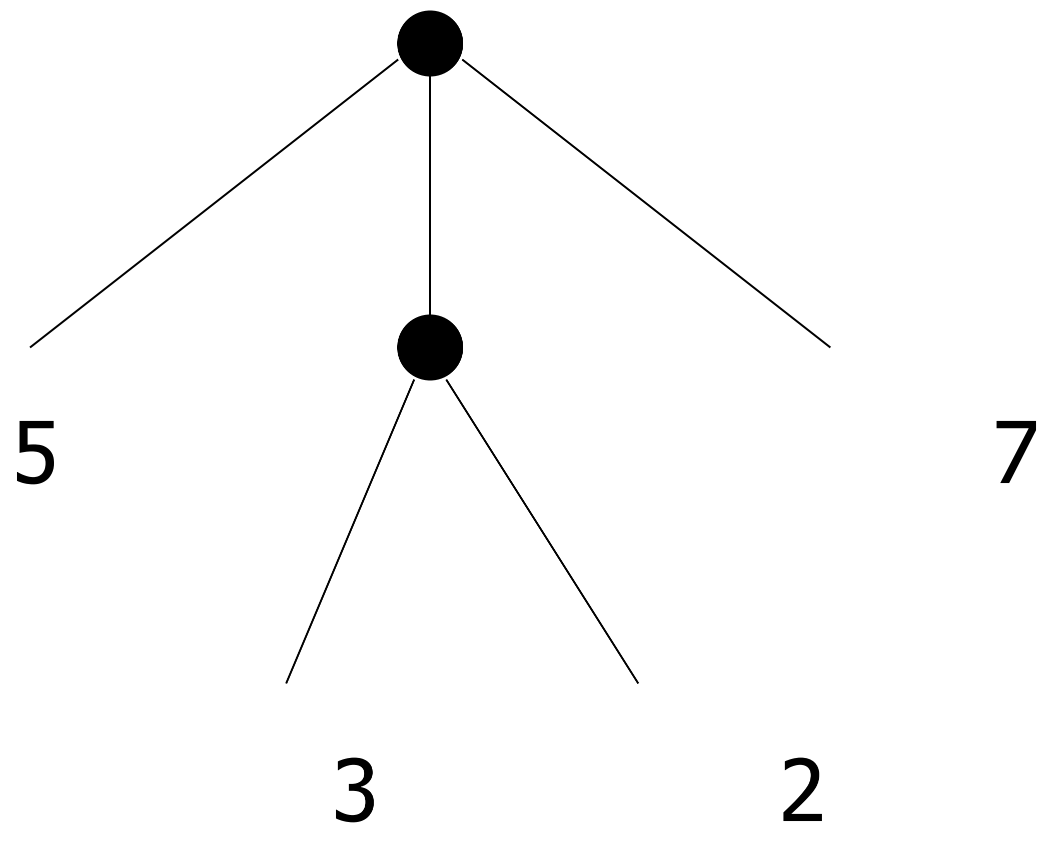
```
val tree : int tree =
 TreeNode
  (More
    (TreeLeaf 5,
     More
       (TreeNode (More (TreeLeaf 3,
        Last (TreeLeaf 2))),
   Last (TreeLeaf 7))))
```

Mutually Recursive Types - Values



Mutually Recursive Types - Values

A more conventional picture



Mutually Recursive Functions

```
# let rec fringe tree =
    match tree with
       (TreeLeaf x) -> [x]
  | (TreeNode list) -> list_fringe list
and list_fringe tree_list =
    match tree_list with
       (Last tree) -> fringe tree
  | (More (tree,list)) ->
       (fringe tree) @ (list_fringe list);;
```

val fringe : 'a tree -> 'a list = <fun>
val list_fringe : 'a treeList -> 'a list = <fun>

Mutually Recursive Functions

```
# fringe tree;;
- : int list = [5; 3; 2; 7]
```

Problem

```
# type 'a tree = TreeLeaf of ’a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size

Problem

```
# type 'a tree = TreeLeaf of ’a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size

```
let rec tree_size t =
      match t with TreeLeaf _ ->
      | TreeNode ts ->
```

Problem

```
# type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size

```
let rec tree_size t =
    match t with TreeLeaf _ -> 1
    | TreeNode ts -> treeList_size  ts
```

Problem

```
# type 'a tree = TreeLeaf of ’a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
```

- Define tree_size and treeList_size

```
let rec tree_size t =
      match t with TreeLeaf _ -> 1
      | TreeNode ts -> treeList_size  ts
and treeList_size ts =
```

Problem

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- Define tree_size and treeList_size

```
let rec tree_size t =
      match t with TreeLeaf _ -> 1
      | TreeNode ts -> treeList_size  ts
and treeList_size ts =
      match ts with Last t ->
      | More t ts' ->
```

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# type 'a tree = TreeLeaf of 'a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
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- Define tree_size and treeList_size

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let rec tree_size t =
    match t with TreeLeaf _ -> 1
    | TreeNode ts -> treeList_size  ts
and treeList_size ts =
    match ts with Last t -> tree_size t
    | More t ts' -> tree_size t + treeList_size ts'
```

Problem

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# type 'a tree = TreeLeaf of ’a | TreeNode of 'a treeList
and 'a treeList = Last of 'a tree | More of ('a tree * 'a treeList);;
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- Define tree_size and treeList_size

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let rec tree_size t =
      match t with TreeLeaf _ -> 1
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```

Nested Recursive Types

```
# type 'a labeled_tree =
 TreeNode of ('a * 'a labeled_tree list);;
```

type 'a labeled_tree = TreeNode of ('a * 'a labeled_tree list)

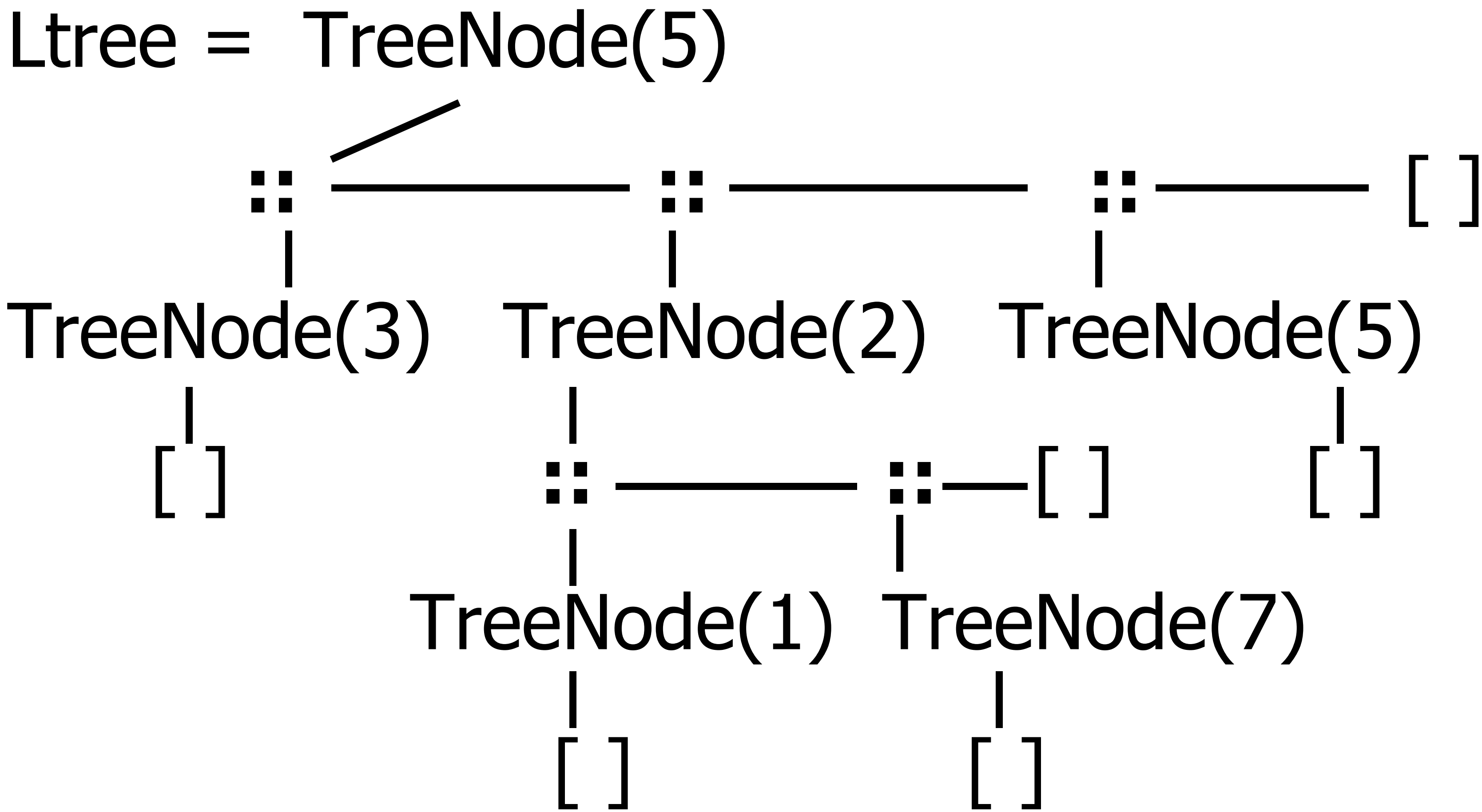
Compare:

```
# type 'a tree =
        TreeLeaf of 'a
      | TreeNode of 'a treeList
and 'a treeList =
        Last of 'a tree
      | More of ('a tree * 'a treeList);;
```

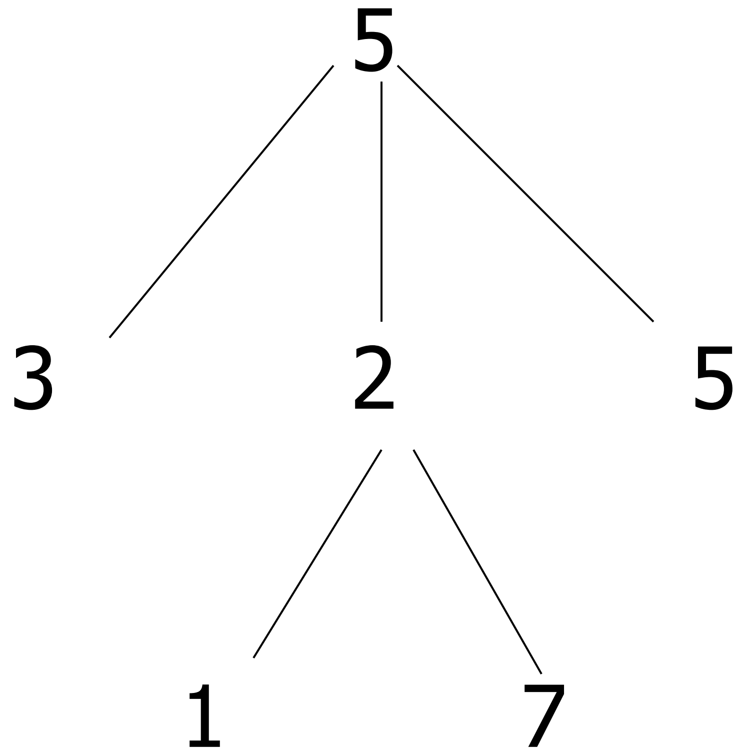
Nested Recursive Type Values

```
# let ltree =
  TreeNode(5,
    [TreeNode (3, []);
     TreeNode (2, [TreeNode (1, []);
                   TreeNode (7, [])]);
     TreeNode (5, [])]);;
```

Nested Recursive Type Values



Nested Recursive Type Values



Mutually Recursive Functions

```
# let rec flatten_tree labtree =
    match labtree with
       TreeNode (x,treelist) ->
            x::flatten_tree_list treelist

  and flatten_tree_list treelist =
     match treelist with
       [] -> []
     | labtree::labtrees ->
         flatten_tree labtree
              @ (flatten_tree_list labtrees);;
```

Mutually Recursive Functions

```
val flatten_tree : 'a labeled_tree -> 'a list = <fun>
val flatten_tree_list : 'a labeled_tree list -> 'a list =
   <fun>
```

```
# flatten_tree ltree;;
-  : int list = [5; 3; 2; 1; 7; 5]
```

- **Nested recursive types lead to mutually recursive functions**