Programming Languages and Compilers (CS 421)

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https://courses.engr.illinois.edu/cs421/fa2024/CS421C

Based on slides by Elsa Gunter, which are based in part on previous slides by Mattox Beckman and updated by Vikram Adve and Gul Agha

Two concepts we learned

- **Immutable state**
  - Variables don’t vary
  - Environment maps variable names to values
  - Closures for functions
- **Functions are first-class values in programs**
  - We can treat them as code (apply)
  - We can treat them as data: pass as function arguments, store in variables, or return as results

+ We’ve seen OCAML has an unintuitive syntax

More Literature on Ocaml

- Book from Harvard Course:
  - https://book.cs51.io//pdfs/abstraction.pdf
  - See first 11 chapters, and later chapters on semantics/evaluation
- Book from Cornell Course:
  - https://cs3110.github.io/textbook/cover.html

Recall: Functions

```
# let plus_two n = n + 2;;
val plus_two : int -> int = <fun>
# plus_two 17;;
- : int = 19

# let plus_two = fun n -> n + 2;;
val plus_two : int -> int = <fun>
# plus_two 14;;
- : int = 16
```

First definition syntactic sugar for second

Recall: Using a nameless function

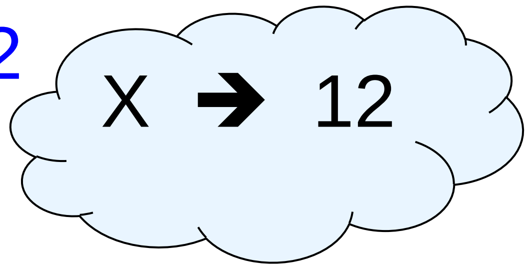
```
# (fun x -> x * 3) 5;;    (* An application *)
- : int = 15

# ((fun y -> y +. 2.0), (fun z -> z * 3));;
  (* As data *)
- : (float -> float) * (int -> int) = (<fun>, <fun>)
```

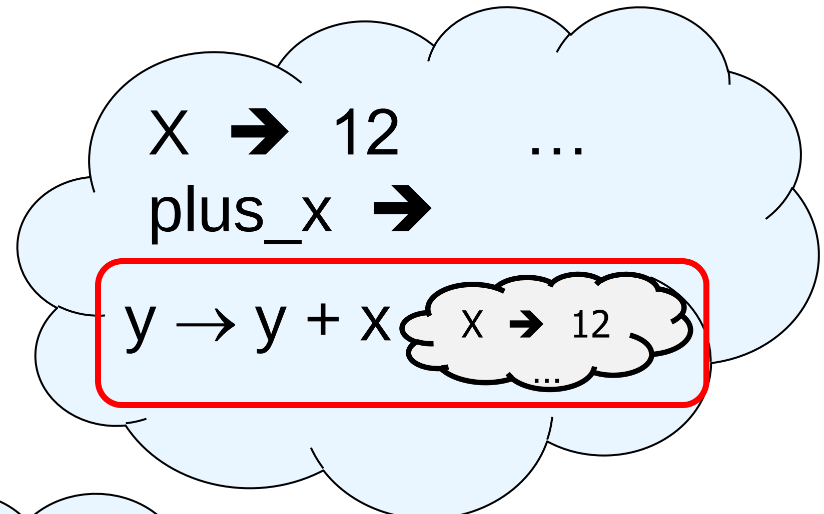
Note: in `fun v -> expression(v)`, the scope of variable is only the body `expression(v)`

Recall: let plus_x = fun x => y + x

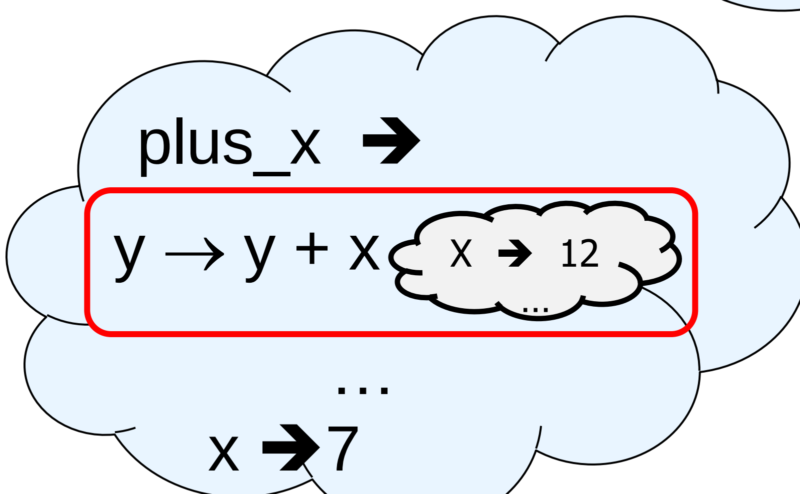
let x = 12



let plus_x = fun y -> y + x



let x = 7



Recall: Functions with more than one argument

```
# let add_three x y z = x + y + z;;
   val add_three : int -> int -> int -> int = <fun>
```

- Remember, it is same as:

```
let add_three =
    fun x -> (fun y -> (fun z -> x + y + z));;
```

- Closure:

```
  < x -> fun y -> (fun z -> x + y + z) , {    } >
```

Free variable ***Return value*** ***Binding***

Functions as arguments

```
# let thrice f x = f (f (f x));;
val thrice : ('a -> 'a) -> 'a -> 'a = <fun>

# let g = thrice plus_two;;
val g : int -> int = <fun>

# g 4;;
- : int = 10

# thrice (fun s -> "Hi! " ^ s) "Good-bye!";;
- : string = "Hi! Hi! Hi! Good-bye!"
```

Tuples

- Pairs:

```
# (1,2)
-: int * int = (1, 2)
```

- And beyond:

```
# (1,2,3,4,5)
- : int * int * int * int * int = (1, 2, 3, 4, 5)
```

Tuples as Values

```
//  ρ0 = {c → 4, a → 1, b → 5}
# let s = (5,"hi",3.2);;
val s : int * string * float = (5, "hi", 3.2)

//  ρ = {s → (5, "hi", 3.2), c → 4, a → 1, b → 5}
```

The size of tuples is fixed!

Access Tuple Elements: Pattern Matching

```
//  ρ = {s → (5, "hi", 3.2), a → 1, b → 5, c → 4}

# let (a,b,c) = s;;        (* (a,b,c) is a pattern *)
val a : int = 5
val b : string = "hi"
val c : float = 3.2

# let (a, _, _) = s;;
val a : int = 5

# let x = 2, 9.3;;      (* tuples don't require parens in Ocaml *)
val x : int * float = (2, 9.3)
```

Nested Tuples

```
#  (*Tuples can be nested *)
# let d = ((1,4,62),("bye",15),73.95);;
val d : (int * int * int) * (string * int) * float =
  ((1, 4, 62), ("bye", 15), 73.95)

#  (*Patterns can be nested *)
# let (p, (st,_), _) = d;;
                 (* _ matches all, binds nothing *)
val p : int * int * int = (1, 4, 62)
val st : string = "bye"
```

First Neuron in OCAML

```
let weights = (1.1, 0.1, 0.5) ;;

let relu x =
    if x > 0.0 then x else 0.0 ;;

let neuron w (x1, x2, x3) =     # weigths and inputs
    let (w1, w2, w3) = w in
    let mult = w1 *. x1 +. w2 *. x2 +. w3 *. x3  in
    relu mult ;;

neuron weights (1.0, 1.1, 2.0) ;;   # returns: 2.21
neuron weights (-1.0, 1.1, -2.0) ;; # returns: 0.0
```

Match Expressions

```
# let triple_to_pair triple =
  match triple
  with (0, x, y) -> (x, y)
  | (x, 0, y) -> (x, y)
  | (x, y, _) -> (x, y);;

val triple_to_pair :
       int * int * int -> int * int = <fun>
```

- Each clause: pattern on the left, expression on the right
- Each x&y have scope of only this clause
- Use the first matching clause

If-then-else as a Match

- Special case of pattern match:

```
# if test then expr1 else expr2
```

- Same as

```
# match test with
    true -> expr1
  | false -> expr2
```

Functions on tuples

```
# let plus_pair (n,m) = n + m;;
val plus_pair : int * int -> int = <fun>

# plus_pair (3,4);;
- : int = 7

# let twice x = (x,x);;
val twice : 'a -> 'a * 'a = <fun>

# twice 3;;
- : int * int = (3, 3)

# twice "hi";;
- : string * string = ("hi", "hi")
```

Curried vs Uncurried

- Recall

```
# let add_three u v w = u + v + w;;
val add_three : int -> int -> int -> int = <fun>
```

  - `let add_three = fun x -> (fun y -> (fun z -> x + y +z ))`

- How does it differ from

```
# let add_triple (u,v,w) = u + v + w;;
val add_triple : int * int * int -> int = <fun>
```

- add_three is **curried**;
- add_triple is **uncurried**

Curried vs Uncurried

```
# add_three 6 3 2;;
- : int = 11

# add_triple (6,3,2);;
- : int = 11

# add_triple 5 4;;
Characters 0-10:  add_triple 5 4;;
                      ^^^^^^^^^^
This function is applied to too many arguments,
maybe you forgot a `;'

# fun x -> add_triple (5,4,x);;
: int -> int = <fun>
```

Save the Environment!

- A *closure* is a pair of an environment and an association of a pattern (e.g. (v1,…,vn) giving the input variables) with an expression (the function body), written:

$$< (v1,\dots,vn) \rightarrow exp, \rho >$$

- Where ρ is the environment in effect when the function is defined (for a simple function)

Closure for plus_pair

- Assume ρ_{plus_pair} was the environment just before plus_pair defined
- Closure for fun (n,m) -> n + m:

$$<(n,m) \rightarrow n + m, \rho_{plus\_pair}>$$

- Environment just after plus_pair defined:

$$\{plus\_pair \rightarrow <(n,m) \rightarrow n + m, \rho_{plus\_pair}>\}$$

$$+ \rho_{plus\_pair}$$

Next: Evaluation Rules

Precisely state how program's code creates new environment

- Declarations
- Expressions

```
Eval ( expr , start_env) -> new_env
```

- Code is a transformer of environments
  - There are no in-place modifications!

Evaluating declarations

- Evaluation uses an environment ρ
- To evaluate a (simple) declaration let x = e
  - Evaluate expression e in ρ to value v
  - Update ρ with $x \rightarrow v$: $\{x \rightarrow v\} + \rho$

Evaluating declarations

- Evaluation uses an environment ρ
- To evaluate a (simple) declaration let x = e
  - Evaluate expression e in ρ to value v
  - Update ρ with x v: $\{x \to v\} + \rho$

Definition of $\rho_1 + \rho_2$

- **Update:** $\rho_1 + \rho_2$ has all the bindings in ρ_1 and all those in ρ_2 that are not rebound in ρ_1

$\{x \to 2, y \to 3, a \to \text{“hi”}\} + \{y \to 100, b \to 6\}$

$= \{x \to 2, y \to 3, a \to \text{“hi”}, b \to 6\}$

Evaluating expressions in OCaml

- Evaluation uses an environment ρ
- A constant evaluates to itself, including primitive operators like + and =

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- Evaluation uses an environment ρ
- A constant evaluates to itself, including primitive operators like + and =
- To evaluate a variable, look it up in ρ: $\rho(v)$
- To evaluate a tuple $(e_1,\ldots,e_n)$,
  - Evaluate each e_i to v_i, right to left for Ocaml
  - Then make value $(v_1,\ldots,v_n)$

Evaluating expressions in OCaml

- To evaluate uses of +, - , etc, eval args, (right to left for Ocaml)[1], then do operation

[1] For discussion why see Xaver Leroy's MS thesis https://xavierleroy.org/publi/ZINC.pdf (Sec 2.2.3)

Evaluating expressions in OCaml

- To evaluate uses of +, - , etc, eval args, (right to left for Ocaml), then do operation
- Function expression evaluates to its closure

Evaluating expressions in OCaml

- To evaluate uses of +, - , etc, eval args, (right to left for Ocaml), then do operation

- Function expression evaluates to its closure

  fun x -> e1

- To evaluate a local dec: let x = e1 in e2
  - Eval e1 to v, then eval e2 using $\{x \rightarrow v\} + \rho$

Evaluating expressions in OCaml

- To evaluate uses of +, - , etc, eval args (right to left for Ocaml), then do operation
- Function expression evaluates to its closure
- To evaluate a local dec: let x = e1 in e2
  - Eval e1 to v, then eval e2 using $\{x \rightarrow v\} + \rho$
- To evaluate a conditional expression: if b then e1 else e2
  - Evaluate b to a value v
  - If v is True, evaluate e1
  - If v is False, evaluate e2

Evaluation of Application with Closures

- Given application expression f argexpr
- In Ocaml, evaluate argexpr to value v
- In environment ρ, evaluate left term f to closure, $c = <(x_1,\ldots,x_n) \rightarrow bodyexpr, \rho'>$
  - $(x_1,\ldots,x_n)$ variables in (first) argument
  - v must have form $(v_1,\ldots,v_n)$
- Update the environment ρ' to

  $\rho'' = \{x_1 \rightarrow v_1,\ldots, x_n \rightarrow v_n\}+ \rho'$
- Evaluate body bodyexpr in environment ρ''

Application of functions

let f x = x + 1;;

f 3

Declaration is same as:

let f = fun x -> f x

1. Assume starting environment: { }

Evaluation of decl: f-> <x -> x + 1, {}>

Evaluation of application (f 3):

x + 1, {x->3} ====> 4

1. Assume starting environment: { x->5 }

Evaluation of decl: f-> <x -> x + 1, {x->5}>

Evaluation of application (f 3):

x + 1, {x->3} ====> 4 [recall how we computed ρ'']

Application of functions

```
let add_three x y z = x + y + z;;
# add_three 5 4 3;;
-: int -> int = <fun>


Why?
 (fun x -> (fun y -> (fun z -> x + y + z)) ) 5 4 3
Evaluating the environments ():
  (<x -> (fun y -> (fun z -> x + y + z)), { }>) 5 4 3
  (<y -> (fun z -> x + y + z), {x->5})> ) 4 3
  (<z -> x + y + z, {x->5, y->4}) 3
  ( x + y + z, {x ->5, y->4, z->3} )
  ( 12 )
```

Finally, a simple expression

Eager vs Lazy Evaluation

- Given application expression f argexpr
- In Ocaml, evaluate argexpr to value v
- In environment ρ, evaluate left term to closure, $c = <(x_1,\ldots,x_n) \rightarrow b, \rho'>$
- …
- Evaluate body bodyexpr in environment ρ''

This is **eager evaluation**!

In contrast, **lazy evaluation** would evaluate argexpr only when (or if!) its result is needed in bodyexpr!

- OCAML has eager evaluation, Haskell has lazy.

Eager vs Lazy Evaluation

Sample:

```
1: let x = 1;
2: fun f test val =
3:    if test then val + 1
4:    else 0
```

Eager

`f true (x+1)`

- Evaluates x+1 immediately, calls f true 2

`f false (x+1)`

- Same

`f false (expensive x)`

- Computes unnecessary expensive result

Lazy

- calls f but delays evaluating the expression until line 3;
- Doesn't evaluate x+1 at all (because test = false, goes to else)
- Does not execute the result as it is not needed

Extra Material for Extra Credit

Evaluating expressions in OCaml

- Evaluation uses an environment ρ
  - Eval (e , ρ)
- A constant evaluates to itself, including primitive operators like + and =
  - Eval (c , ρ) => Val c
- To evaluate a variable v, look it up in ρ:
  - Eval (v, ρ) => Val (ρ(v))

Evaluating expressions in OCaml

- To evaluate a tuple $(e_1,\ldots,e_n)$,
  - Evaluate each e_i to v_i, right to left for Ocaml
  - Then make value $(v_1,\ldots,v_n)$
  - $\mathrm{Eval}((e_1,\ldots,e_n),\rho) \Rightarrow \mathrm{Eval}((e_1,\ldots,\mathrm{Eval}\ (e_n, \rho)), \rho)$
  - $\mathrm{Eval}((e_1,\ldots,e_i, \mathrm{Val}\ v_{i+1},\ldots, \mathrm{Val}\ v_n)\ ,\ \rho) \Rightarrow$ $\mathrm{Eval}((e_1,\ldots,\mathrm{Eval}(e_i, \rho), \mathrm{Val}\ v_{i+1},\ldots, \mathrm{Val}\ v_n)\ ,\ \rho)$
  - $\mathrm{Eval}((\mathrm{Val}\ v_1,\ldots,\mathrm{Val}\ v_n)\ ,\ \rho) \Rightarrow \mathrm{Val}\ (v_1,\ldots,v_n)$

Evaluating expressions in OCaml

- To evaluate uses of +, - , etc, eval args, then do operation ⦿ (+, -, *, +., ….)
  - Eval(e_1⦿e_2, ρ) => Eval(e_1⦿Eval(e_2, ρ), ρ))
  - Eval(e_1⦿Val e_2, ρ)=>Eval(Eval(e_1, ρ)⦿Val v_2, ρ))
  - Eval(Val v_1⦿ Val v_2) => Val (v_1⦿ v_2)

- Function expression evaluates to its closure
  - Eval (fun x -> e, ρ) => Val < x -> e, ρ>

Evaluating expressions in OCaml

- To evaluate a local dec: let x = e1 in e2
  - Eval e1 to v, then eval e2 using $\{x \rightarrow v\} + \rho$
  - Eval(let x = e_1 in e_2, ρ) =>
    Eval(let x = Eval(e_1, ρ) in e_2, ρ)
  - Eval(let x = Val v in e_2, ρ) =>
    Eval(e_2, $\{x \rightarrow v\} + \rho$)

Evaluating expressions in OCaml

- To evaluate a conditional expression:
  if b then e_1 else e_2
  - Evaluate b to a value v
  - If v is True, evaluate e_1
  - If v is False, evaluate e_2

  - Eval(if b then e_1 else e_2, ρ) =>
    Eval(if Eval(b, ρ) then e_1 else e_2, ρ)
  - Eval(if Val true then e_1 else e_2, ρ) =>Eval(e_1, ρ)
  - Eval(if Val false then e_1 else e_2, ρ) =>Eval(e_2, ρ)

Evaluation of Application with Closures

- Given application expression $f\ e$
- In Ocaml, evaluate e to value v
- In environment ρ, evaluate left term to closure, $c = <(x_1,\ldots,x_n) \rightarrow b, \rho'>$
  - $(x_1,\ldots,x_n)$ variables in (first) argument
  - v must have form $(v_1,\ldots,v_n)$
- Update the environment ρ' to

  $\rho'' = \{x_1 \rightarrow v_1,\ldots, x_n \rightarrow v_n\} + \rho'$
- Evaluate body b in environment ρ''

Evaluation of Application with Closures

- $\text{Eval}(f\ e, \rho) \Rightarrow \text{Eval}(f\ (\text{Eval}(e, \rho)), \rho)$

- $\text{Eval}(f\ (\text{Val}\ v), \rho) \Rightarrow \text{Eval}((\text{Eval}(f, \rho))\ (\text{Val}\ v), \rho)$

- $\text{Eval}((\text{Val} <(x_1,\ldots,x_n) \rightarrow b, \rho'>)(\text{Val}\ (v_1,\ldots,v_n)), \rho) \Rightarrow$
  $\text{Eval}(b, \{x_1 \rightarrow v_1,\ldots, x_n \rightarrow v_n\}+\rho')$

Evaluation of Application of plus_x;;

- Have environment:

  ρ = {plus_x → <y → y + x, ρ_{plus_x} >, … ,
  y → 19, x →17, z →3, …}

  where ρ_{plus_x} = {x → 12, … , y → 24, …}
- Eval (plus_x z, ρ) =>
- Eval(plus_x (Eval(z, ρ))) => …

Evaluation of Application of plus_x;;

- Have environment:

  ρ = {plus_x → <y → y + x, ρ_{plus_x} >, … ,
  y → 19, x →17, z →3, …}

  where ρ_{plus_x} = {x → 12, … , y → 24, …}
- Eval (plus_x z, ρ) =>
- Eval(plus_x (Eval(z, ρ)), ρ) =>
- Eval(plus_x (Val 3), ρ) => …

Evaluation of Application of plus_x;;

- Have environment:

  ρ = {plus_x $\rightarrow$ <y $\rightarrow$ y + x, ρ_{plus_x} >, … ,
  y $\rightarrow$ 19, x $\rightarrow$ 17, z $\rightarrow$ 3, …}

  where ρ_{plus_x} = {x $\rightarrow$ 12, … , y $\rightarrow$ 24, …}
- Eval (plus_x z, ρ) =>
- Eval (plus_x (Eval(z, ρ)), ρ) =>
- Eval (plus_x (Val 3), ρ) =>
- Eval ((Eval(plus_x, ρ)) (Val 3), ρ) => …

Evaluation of Application of plus_x;;

- Have environment:

  $\rho = \{plus\_x \rightarrow \langle y \rightarrow y + x, \rho_{plus\_x} \rangle, \ldots ,$
  $y \rightarrow 19, x \rightarrow 17, z \rightarrow 3, \ldots\}$

  where $\rho_{plus\_x} = \{x \rightarrow 12, \ldots , y \rightarrow 24, \ldots\}$
- Eval (plus_x z, ρ) =>
- Eval (plus_x (Eval(z, ρ)), ρ) =>
- Eval (plus_x (Val 3), ρ) =>
- Eval ((Eval(plus_x, ρ)) (Val 3), ρ) =>
- Eval ((Val$\langle y \rightarrow y + x, \rho_{plus_x} \rangle$)(Val 3), ρ) => …

Evaluation of Application of plus_x;;

- Have environment:

  ρ = {plus_x $\rightarrow$ <y $\rightarrow$ y + x, ρ_{plus_x} >, … ,
  y $\rightarrow$ 19, x $\rightarrow$17, z $\rightarrow$3, …}

  where ρ_{plus_x} = {x $\rightarrow$ 12, … , y $\rightarrow$ 24, …}
- Eval ((Val<y $\rightarrow$ y + x, ρ_{plus_x} >)(Val 3), ρ) => …

Evaluation of Application of plus_x;;

- Have environment:

  $\rho = \{plus\_x \rightarrow <y \rightarrow y + x, \rho_{plus\_x} >, \ldots ,$
  $y \rightarrow 19, x \rightarrow 17, z \rightarrow 3, \ldots\}$

  where $\rho_{plus\_x} = \{x \rightarrow 12, \ldots , y \rightarrow 24, \ldots\}$
- Eval ((Val<$y \rightarrow y + x, \rho_{plus\_x}$ >)(Val 3), ρ) =>
- Eval ($y + x, \{y \rightarrow 3\} + \rho_{plus\_x}$) => …

Evaluation of Application of plus_x;;

- Have environment:

  ρ = {plus_x → <y → y + x, ρ_{plus_x} >, … ,
  y → 19, x →17, z →3, …}

  where ρ_{plus_x} = {x → 12, … , y → 24, …}
- Eval ((Val<y → y + x, ρ_{plus_x} >)(Val 3), ρ) =>
- Eval (y + x, {y → 3} +ρ_{plus_x}) =>
- Eval(y+Eval(x, {y → 3} +ρ_{plus_x}),
  {y → 3} +ρ_{plus_x}) => …

Evaluation of Application of plus_x;;

- Have environment:

  ρ = {plus_x → <y → y + x, ρ_{plus_x} >, … ,
  y → 19, x →17, z →3, …}

  where ρ_{plus_x} = {x → 12, … , y → 24, …}
- Eval ((Val<y → y + x, ρ_{plus_x} >)(Val 3), ρ) =>
- Eval (y + x, {y → 3} +ρ_{plus_x}) =>
- Eval(y+Eval(x, {y → 3} +ρ_{plus_x}),
  {y → 3} +ρ_{plus_x}) =>
- Eval(y+Val 12,{y → 3} +ρ_{plus_x}) => …

Evaluation of Application of plus_x;;

- Have environment:

  $\rho = \{plus\_x \to <y \to y + x, \rho_{plus\_x}>, \ldots,$
  $y \to 19, x \to 17, z \to 3, \ldots\}$

  where $\rho_{plus\_x} = \{x \to 12, \ldots, y \to 24, \ldots\}$
- $Eval(y+Eval(x, \{y \to 3\} + \rho_{plus\_x}),$
  $\{y \to 3\} + \rho_{plus\_x}) =>$
- $Eval(y+Val\ 12, \{y \to 3\} + \rho_{plus\_x}) =>$
- $Eval(Eval(y, \{y \to 3\} + \rho_{plus\_x}) +$
  $Val\ 12, \{y \to 3\} + \rho_{plus\_x}) =>\ldots$

Evaluation of Application of plus_x;;

- Have environment:

  $\rho = \{plus\_x \rightarrow \langle y \rightarrow y + x, \rho_{plus\_x} \rangle, \ldots ,$
  $y \rightarrow 19, x \rightarrow 17, z \rightarrow 3, \ldots\}$

  where $\rho_{plus\_x} = \{x \rightarrow 12, \ldots , y \rightarrow 24, \ldots\}$
- Eval(Eval(y, $\{y \rightarrow 3\} + \rho_{plus\_x}$) +
  Val 12, $\{y \rightarrow 3\} + \rho_{plus\_x}$) =>
- Eval(Val 3 + Val 12 , $\{y \rightarrow 3\} + \rho_{plus\_x}$) =>…

Evaluation of Application of plus_x;;

- Have environment:

  $\rho = \{plus\_x \to \langle y \to y + x, \rho_{plus\_x} \rangle, \ldots ,$
  $y \to 19, x \to 17, z \to 3, \ldots\}$

  where $\rho_{plus\_x} = \{x \to 12, \ldots , y \to 24, \ldots\}$
- $Eval(Eval(y, \{y \to 3\} + \rho_{plus\_x}) +$
  $Val\ 12, \{y \to 3\} + \rho_{plus\_x}) \Rightarrow$
- $Eval(Val\ 3 + Val\ 12, \{y \to 3\} + \rho_{plus\_x}) \Rightarrow$
- $Val\ (3 + 12) = Val\ 15$

Evaluation of Application of plus_pair

- Assume environment

$\rho = \{x \rightarrow 3\ldots,$

$\quad \text{plus\_pair} \rightarrow <(n,m) \rightarrow n + m, \rho_{\text{plus\_pair}}>\} + \rho_{\text{plus\_pair}}$

- Eval (plus_pair (4,x), ρ)=>
- Eval (plus_pair (Eval ((4, x), ρ)), ρ) =>
- Eval (plus_pair (Eval ((4, Eval (x , ρ)), ρ)), ρ) =>
- Eval (plus_pair (Eval ((4, Val 3), ρ)), ρ) =>
- Eval (plus_pair (Eval ((Eval (4, ρ), Val 3), ρ)), ρ) =>
- Eval (plus_pair (Eval ((Val 4, Val 3), ρ)), ρ) =>

Evaluation of Application of plus_pair

- Assume environment

$\rho = \{x \to 3\ldots,$
$\quad plus\_pair \to <(n,m) \to n+m, \rho_{plus\_pair}>\} + \rho_{plus\_pair}$

- Eval (plus_pair (Eval ((Val 4, Val 3), ρ)), ρ) =>
- Eval (plus_pair (Val (4, 3)), ρ) =>
- Eval (Eval (plus_pair, ρ), Val (4, 3)), ρ) => ...
- Eval ((Val<(n,m)$\to$n+m, ρ_{plus_pair}>)(Val(4,3)) , ρ)=>
- Eval (n + m, {n -> 4, m -> 3} + ρ_{plus_pair}) =>
- Eval (4 + 3, {n -> 4, m -> 3} + ρ_{plus_pair}) => 7

Closure question

- If we start in an empty environment, and we execute:

```
let f = fun n -> n + 5;;
(* 0 *)
let pair_map g (n,m) = (g n, g m);;
let f = pair_map f;;
let a = f (4,6);;
```

What is the environment at (* 0 *)?

Answer

let f = fun n -> n + 5;;

$\rho_0 = \{f \to \langle n \to n + 5, \{\ \}\rangle\}$

Closure question

- If we start in an empty environment, and we execute:

```
let f = fun => n + 5;;
let pair_map g (n,m) = (g n, g m);;
(* 1 *)
let f = pair_map f;;
let a = f (4,6);;
```

What is the environment at (* 1 *)?

Answer

ρ_0 = {f → <n → n + 5, { }>}

let pair_map g (n,m) = (g n, g m);;

ρ_1 = {pair_map →

<g → fun (n,m) -> (g n, g m),

{f → <n → n + 5, { }>}>,

f → <n → n + 5, { }>}

Closure question

- If we start in an empty environment, and we execute:

```
let f = fun => n + 5;;
let pair_map g (n,m) = (g n, g m);;
let f = pair_map f;;
(* 2 *)
let a = f (4,6);;
```

What is the environment at (* 2 *)?

Evaluate pair_map f

$\rho_0 = \{f \rightarrow <n \rightarrow n + 5, \{\ \}>\}$

$\rho_1 = \{pair\_map \rightarrow <g \rightarrow fun\ (n,m)\ \text{->}\ (g\ n, g\ m), \rho_0>,$

$\qquad f \rightarrow <n \rightarrow n + 5, \{\ \}>\}$

let f = pair_map f;;

Evaluate pair_map f

ρ_0 = {f → <n → n + 5, { }>}

ρ_1 = {pair_map →<g→fun (n,m) -> (g n, g m), ρ_0>,
f→<n → n + 5, { }>}

Eval(pair_map f, ρ_1) =

Evaluate pair_map f

ρ_0 = {f → <n → n + 5, { }>}

ρ_1 = {pair_map →<g→fun (n,m) -> (g n, g m), ρ_0>,
f→<n → n + 5, { }>}

Eval(pair_map f, ρ_1) =>

Eval(pair_map (Eval(f, ρ_1)), ρ_1) =>

Eval(pair_map (Val<n → n + 5, { }>), ρ_1) =>

Eval((Eval(pair_map, ρ_1))(Val<n → n+5, { }>), ρ_1) =>

Eval((Val (<g→fun (n,m) -> (g n, g m), ρ_0>)
(Val <n → n + 5, { }>), ρ_1) =>

Eval(fun (n,m)->(g n, g m), {g→<n→n + 5, { }>}+ρ_0)
=>

Evaluate pair_map f

$\rho_0 = \{f \to <n \to n + 5, \{\ \}>\}$

$\rho_1 = \{pair\_map \to <g \to fun\ (n,m) \text{ -> } (g\ n, g\ m), \rho_0>,$

$\qquad f \to <n \to n + 5, \{\ \}>\}$

$Eval(pair\_map\ f, \rho_1) => \ldots =>$

$Eval(fun\ (n,m)\text{->}(g\ n, g\ m), \{g \to <n \to n + 5, \{\ \}>\} + \rho_0)$

$=$

$Eval(fun\ (n,m)\text{->}(g\ n, g\ m),$

$\qquad \{g \to <n \to n + 5, \{\ \}>, f \to <n \to n + 5, \{\ \}>\}) =>$

$Val\ (<(n,m) \to (g\ n, g\ m),$

$\qquad \{g \to <n \to n + 5, \{\ \}>, f \to <n \to n + 5, \{\ \}>\})$

Answer

ρ_1 = {pair_map →
<g → fun (n,m) -> (g n, g m),{f → <n → n + 5, { }>}>,
f → <n → n + 5, { }>}

let f = pair_map f;;

ρ_2 = {f → <(n,m) →(g n, g m),
{g → <n → n + 5, { }>,
f → <n → n + 5, { }>}>,
pair_map → <g → fun (n,m) -> (g n, g m),
{f → <n → n + 5, { }>}>}

(*Remember: the original f is now removed from ρ_2 *)

Closure question

- If we start in an empty environment, and we execute:

```
let f = fun => n + 5;;
let pair_map g (n,m) = (g n, g m);;
let f = pair_map f;;
let a = f (4,6);;
```

(* 3 *)

What is the environment at (* 3 *)?

Final Evalution?

ρ_2 = {f → <(n,m) →(g n, g m),
{g → <n → n + 5, { }>,
f → <n → n + 5, { }>}>,
pair_map → <g → fun (n,m) -> (g n, g m),
{f → <n → n + 5, { }>}>}

let a = f (4,6);;

Evaluate f (4,6);;

ρ_2 = {f → <(n,m) →(g n, g m),

{g → <n → n + 5, { }>,

f → <n → n + 5, { }>}>,

pair_map → <g → fun (n,m) -> (g n, g m),

{f → <n → n + 5, { }>}>}

Eval(f (4,6), ρ_2) =

Evaluate f (4,6);;

$\rho_2 = \{f \rightarrow <(n,m) \rightarrow (g\ n,\ g\ m),$

$\{g \rightarrow <n \rightarrow n + 5, \{\ \}>,$

$f \rightarrow <n \rightarrow n + 5, \{\ \}>\}>,$

$\text{pair\_map} \rightarrow <g \rightarrow \text{fun } (n,m) \text{ -> } (g\ n,\ g\ m),$

$\{f \rightarrow <n \rightarrow n + 5, \{\ \}>\}>\}$

Eval(f (4,6), ρ_2) => Eval(f (Eval((4,6), ρ_2)), ρ_2) =>

Eval(f (Eval((4,Eval(6, ρ_2)), ρ_2)), ρ_2) =>

Eval(f (Eval((4,Val 6), ρ_2)), ρ_2) =>

Eval(f (Eval((Eval(4, ρ_2),Val 6), ρ_2)), ρ_2) =>

Eval(f (Eval((Val 4,Val 6), ρ_2)), ρ_2) =>

Evaluate f (4,6);;

ρ_2 = {f → <(n,m) →(g n, g m),
{g → <n → n + 5, { }>,
f → <n → n + 5, { }>}>,
pair_map → <g → fun (n,m) -> (g n, g m),
{f → <n → n + 5, { }>}>}

Eval(f (4,6), ρ_2) => … =>

Eval(f (Eval((Val 4,Val 6), ρ_2)), ρ_2) =>

Eval(f (Val (4, 6)), ρ_2) =>

Eval(Eval(f, ρ_2) (Val (4, 6)), ρ_2) =>

Evaluate f (4,6);;

ρ_2 = {f → <(n,m) →(g n, g m),
{g → <n → n + 5, { }>,
f → <n → n + 5, { }>}>,
pair_map → <g → fun (n,m) -> (g n, g m),
{f → <n → n + 5, { }>}>}

Eval(f (4,6), ρ_2) => … =>

Eval(Eval(f, ρ_2) (Val (4, 6)), ρ_2) =>

Eval((Val <(n,m)→(g n, g m),
{g→<n→n+5, { }>,
f→<n→n+5, { }>}>)(Val(4,6)))),ρ_2)=>

Evaluate f (4,6);;

ρ_2 = {f → <(n,m) →(g n, g m),
{g → <n → n + 5, { }>,
f → <n → n + 5, { }>}>,
pair_map → <g → fun (n,m) -> (g n, g m),
{f → <n → n + 5, { }>}>}

Eval((Val <(n,m)→(g n, g m),
{g→<n→n+5, { }>,
f→<n→n+5, { }>}>)(Val(4,6)))),ρ_2)=>

Eval((g n, g m), {n → 4, m → 6, g→<n→n+5, { }>,
f→<n→n+5, { }>}) =>

Evaluate f (4,6);;

Let ρ′ = {n → 4, m → 6, g→<n→n+5, { }>,
f→<n→n+5, { }>})

Eval((g n, g m), {n → 4, m → 6, g→<n→n+5, { }>,
f→<n→n+5, { }>}) =

Eval((g n, g m), ρ′) =>

Eval((g n, Eval(g m, ρ′)), ρ′) =>

Eval((g n, Eval(g (Eval (m, ρ′)), ρ′)), ρ′) =>

Eval((g n, Eval(g (Val 6), ρ′)), ρ′) =>

Eval((g n, Eval((Eval(g, ρ′))(Val 6), ρ′)), ρ′) =>

Evaluate f (4,6);;

Let ρ′ = {n → 4, m → 6, g→<n→n+5, { }>,

f→<n→n+5, { }>})

Eval((g n, Eval((Eval(g, ρ′))(Val 6), ρ′)), ρ′) =>

Eval((g n, Eval((Val<n→n+5,{ }>)(Val 6), ρ′)), ρ′) =>

Eval((g n, Eval(n+5, {n→6}+{ })), ρ′) =

Eval((g n, Eval(n+5, {n→6})), ρ′) =>

Eval((g n, Eval(n+(Eval(5, {n→6})), {n→6})), ρ′) =>

Eval((g n, Eval(n+(Val 5), {n→6})), ρ′) =>

Eval((g n, Eval((Eval(n,{n→6}))+(Val 5),{n→6}), ρ′)=>

Eval((g n, Eval((Val 6)+(Val 5),{n→6}), ρ′)=>

Evaluate f (4,6);;

Let ρ′ = {n → 4, m → 6, g→<n→n+5, { }>,
f→<n→n+5, { }>})

Eval((g n, Eval((Val 6)+(Val 5),{n→6}), ρ′) =>

Eval((g n, Val 11), ρ′) =>

Eval((Eval(g n, ρ′), Val 11), ρ′) =>

Eval((Eval(g (Eval(n, ρ′)), ρ′), Val 11), ρ′) =>

Eval((Eval(g (Val 4), ρ′), Val 11), ρ′) =>

Eval((Eval(Eval(g, ρ′)(Val 4), ρ′), Val 11), ρ′) =>

Eval((Eval((Val<n→n+5, { }>)(Val 4), ρ′), Val 11), ρ′) =>

Evaluate f (4,6);;

Let ρ′ = {n → 4, m → 6, g→<n→n+5, { }>,
f→<n→n+5, { }>})

Eval((Eval((Val<n→n+5, { }>)(Val 4), ρ′), Val 11), ρ′)
=>

Eval((Eval(n+5, {n → 4}+{})), Val 11), ρ′) =

Eval((Eval(n+5, {n → 4})), Val 11), ρ′) =>

Eval((Eval(n+Eval(5,{n → 4}),{n → 4}), Val 11),ρ′) =>

Eval((Eval(n+(Val 5),{n → 4}), Val 11),ρ′) =>

Eval((Eval(Eval(n,{n → 4})+(Val 5),{n → 4}),
Val 11),ρ′) =>

End of

Extra Material for Extra Credit

Recursive Functions

```
# let rec factorial n =
    if n = 0 then 1
    else n * factorial (n - 1);;
  val factorial : int -> int = <fun>

# factorial 5;;
- : int = 120

# (* rec  is needed for recursive function
  declarations *)
```

Recursion Example

Compute n^2 recursively using:

$$n^2 = (2 * n - 1) + (n - 1)^2$$

```
# let rec nthsq n =            (* rec for recursion *)
   match n with      (* pattern matching for cases *)
     0 -> 0                    (* base case *)
   | n -> (2 * n -1)           (* recursive case *)
           + nthsq (n -1);;    (* recursive call *)

val nthsq : int -> int = <fun>

# nthsq 3;;
- : int = 9
```

Structure of recursion similar to inductive proof

Recursion and Induction

```
# let rec nthsq n =
    match n with
              0 -> 0   (*Base case!*)
            | n -> (2 * n - 1) + nthsq (n - 1) ;;
```

- Base case is the last case; it stops the computation
- Recursive call must be to arguments that are somehow smaller - must progress to base case
- **if or match must contain the base case (!!!)**
  - Failure of selecting base case **will** cause **non-termination**
  - But the program will crash because it exhausts the stack!

Lists

- First example of a recursive datatype (aka algebraic datatype)

- Unlike tuples, lists are homogeneous in type (all elements same type)

Lists

- List can take one of two forms:
  - **Empty list**, written [ ]
  - **Non-empty list**, written x :: xs
    - x is head element,
    - xs is tail list, :: called "cons"
- How we typically write them (syntactic sugar):
  - **[x]** == x :: [ ]
  - **[x1; x2; …; xn]** == x1 :: x2 :: … :: xn :: [ ]

Lists

```
# let fib5 = [8;5;3;2;1;1];;
val fib5 : int list = [8; 5; 3; 2; 1; 1]

# let fib6 = 13 :: fib5;;
val fib6 : int list = [13; 8; 5; 3; 2; 1; 1]

# (8::5::3::2::1::1::[ ]) = fib5;;
- : bool = true

# fib5 @ fib6;;
- : int list =
        [8; 5; 3; 2; 1; 1; 13; 8; 5; 3; 2; 1; 1]
```

Lists are Homogeneous

```
# let bad_list = [1; 3.2; 7];;
Characters 19-22:
  let bad_list = [1; 3.2; 7];;
                     ^^^
```

This expression has type float but is here used with type int

Question

- Which one of these lists is invalid?

1. [2; 3; 4; 6]
2. [2,3; 4,5; 6,7]
3. [(2.3,4); (3.2,5); (6,7.2)]
4. [["hi"; "there"]; ["wahcha"]; [ ]; ["doin"]]

3 is invalid because of the last pair

Functions Over Lists

```
# let rec double_up list =
    match list with
         [ ] -> [ ]  (* pattern before ->,
                        expression after *)
       | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>


(* fib5 = [8;5;3;2;1;1] *)
# let fib5_2 = double_up fib5;;
val fib5_2 : int list = [8; 8; 5; 5; 3; 3; 2; 2;
  1; 1; 1; 1]
```

Functions Over Lists

```
# let silly = double_up ["hi"; "there"];;
val silly : string list = ["hi"; "hi"; "there";
  "there"]

# let rec poor_rev list =
  match list
  with [] -> []
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>

# poor_rev silly;;
- : string list = ["there"; "there"; "hi"; "hi"]
```

Structural Recursion

- Functions on recursive datatypes (eg lists) tend to be recursive

- Recursion over recursive datatypes generally by structural recursion
  - Recursive calls made to components of structure of the same recursive type
  - Base cases of recursive types stop the recursion of the function

Question: Length of list

- Problem: write code for the length of the list
  - How to start?

```
let length l =
```

Question: Length of list

- Problem: write code for the length of the list
  - How to start?

```
let rec length l =
    match l with
```

Question: Length of list

- Problem: write code for the length of the list
    - What patterns should we match against?

```
let rec length l =
    match l with
```

Question: Length of list

- Problem: write code for the length of the list
  - What patterns should we match against?

```
let rec length l =
    match l with [] ->
     | (a :: bs) ->
```

Question: Length of list

- Problem: write code for the length of the list
  - What result do we give when l is empty?

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) ->
```

Question: Length of list

- Problem: write code for the length of the list
  - What result do we give when l is not empty?

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) ->
```

Question: Length of list

- Problem: write code for the length of the list
  - What result do we give when l is not empty?

```
let rec length l =
    match l with [] -> 0
      | (a :: bs) -> 1 + length bs
```

Same Length

- How can we efficiently answer if two lists have the same length?

Tactics:

- First list is empty: then true if second list is empty else false
- First list in not empty: then if second list empty return false, or otherwise compare whether the sublists (after the first element) have the same length

Same Length

- How can we efficiently answer if two lists have the same length?

```
let rec same_length list1 list2 =
  match list1 with
    [] -> (
           match list2 with [] -> true
                        | (y::ys) -> false
        )
  | (x::xs) -> (
           match list2 with [] -> false
                        | (y::ys) -> same_length xs ys
        )
```

Functions Over Lists

```
# let rec map f list =
  match list with
    [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>

# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]

# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
```

Iterating over lists

```
# let rec fold_left f a list =
  match list with
    [] -> a
  | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list
  -> 'a = <fun>

# fold_left
   (fun () -> print_string)
   ()
   ["hi"; "there"];;
hithere- : unit = ()
```

Iterating over lists

```
# let rec fold_right f list b =
  match list with
    [] -> b
  | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b
  -> 'b = <fun>

# fold_right
    (fun s -> fun () -> print_string s)
    ["hi"; "there"]
    ();;
therehi- : unit = ()
```

Your turn: doubleList : int list -> int list

- Write a function that takes a list of int and returns a list of the same length, where each element has been multiplied by 2

let rec doubleList list =

Your turn: doubleList : int list -> int list

- Write a function that takes a list of int and returns a list of the same length, where each element has been multiplied by 2

```
let rec doubleList list =
  match list
    with [] ->[]
      |  x :: xs -> (2 * x) :: doubleList xs
```

Your turn: doubleList : int list -> int list

- Write a function that takes a list of int and returns a list of the same length, where each element has been multiplied by 2

```
let rec doubleList list =
  match list
   with [] ->[]
    | x :: xs -> (2 * x) :: doubleList xs
```

Higher-Order Functions Over Lists

```
# let rec map f list =
  match list
  with [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]
# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
```

Higher-Order Functions Over Lists

```
# let rec map f list =
  match list
  with [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]
# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
```

Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```
# let doubleList list =
    List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```
# let doubleList list =
    List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

- Same function, but no explicit recursion

Folding Recursion

- Another common form "folds" an operation over the elements of the structure

```
# let rec multList list = match list
  with [ ] -> 1
  | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;4;6];;
- : int = 48
```

- Computes (2 * (4 * (6 * 1)))

Folding Recursion : Length Example

```
# let rec length list = match list
  with [ ] -> 0   (* Nil case *)
  | a :: bs -> 1 + length bs;;  (* Cons case *)
val length : 'a list -> int = <fun>
# length [5; 4; 3; 2];;
- : int = 4
```

- Nil case [ ] is base case, 0 is the base value
- Cons case recurses on component list bs
- What do multList and length have in common?