# Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

# Tail Recursion

- A recursive program is tail recursive if all recursive calls are tail calls
- Tail recursive programs may be optimized to be implemented as loops, thus removing the function call overhead for the recursive calls
- Tail recursion generally requires extra "accumulator" arguments to pass partial results
  - May require an auxiliary function

# Terminology

- Available: A function call that can be executed by the current expression
- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).
  - if (h x) then f x else (x + g x)
  - if (h x) then (fun x -> f x) else (g (x + x))

Not available

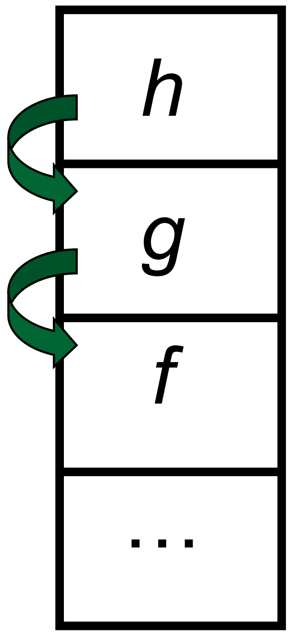
# Terminology

- Tail Position: A subexpression s of expressions e, which is available and such that if evaluated, will be taken as the value of e
  - if (x>3) then $\boxed{\text{x + 2}}$ else $\boxed{\text{x - 4}}$
  - let x = 5 in $\boxed{\text{x + 4}}$
- Tail Call: A function call that occurs in tail position
  - if (h x) then $\boxed{\text{f x}}$ else $\boxed{\text{(x + g x)}}$

# An Important Optimization

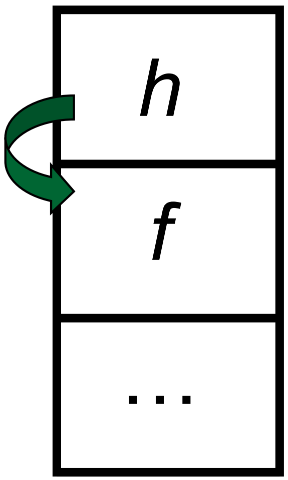
- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?

Normal call



# An Important Optimization

Tail call




- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?
- Then *h* can return directly to *f* instead of *g*

# Tail Recursion - length

- How can we write length with tail recursion?

```
let length list =
   let rec length_aux list acc_length =
         match list
          with [ ] -> acc_length
            | (x::xs) ->
              length_aux xs (1 + acc_length)
   in length_aux list  0
```

# Your turn: list_max – tail recursive

```
#let list_max list =
   let rec max_aux list max_so_far =
      match list with [] ->max_so_far
       | (x :: xs) ->
         max_aux  xs
          (if x > max_so_far then x else max_so_far)
  in
max_aux list (-17)
```

# Your turn: list_max – tail recursive

```
#let list_max list =
 let rec max_aux list curr_max =
    match list with [] -> curr_max
      | (x :: xs) ->
         max_aux xs
           (if x > curr_max then x else curr_max)
  in  (match list
         with [] -> (* ??? *) -1
          | x :: xs -> max_aux xs x)
```

# Iterating over lists

```
# let rec fold_left f a list =
  match list
  with [] -> a
  | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
   <fun>
# fold_left
   (fun () -> print_string)
   ()
   ["hi"; "there"];;
hithere- : unit = ()
```

# Your turn: length, fold_left

let length list =

# Your turn: length, fold_left

```
let length list =
fold_left (fun acc -> fun x -> 1 + acc) list 0
```

# Your turn: list_max – tail recursive

```
#let list_max list =
 let rec max_aux list curr_max =
   match list with [] -> curr_max
     | (x :: xs) ->
        max_aux xs
          (if x > curr_max then x else curr_max)
  in  (match list
        with [] -> (* ??? *) -1
         | x :: xs -> max_aux xs x)
```

# list_max, fold_left

```
let list_max list =
  match list with [] -> (* ??? *) -1
    | (x :: xs) ->
      fold_left
        (fun curr_max -> fun x ->
          if x > curr_max then x else curr_max)
        x
        xs
```

# Folding

```
# let rec fold_left f a list = match list
  with [] -> a | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
   <fun>
```

fold_left f a [$x_1$; $x_2$;…;$x_n$] = f(…(f (f a $x_1$) $x_2$)…)$x_n$

```
# let rec fold_right f list b = match list
  with [ ] -> b | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
   <fun>
```

fold_right f [$x_1$; $x_2$;…;$x_n$] b = f $x_1$(f $x_2$ (…(f $x_n$ b)…))

# Folding

- Can replace recursion by fold_right in any forward primitive recursive definition
  - Primitive recursive means it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by fold_left in any tail primitive recursive definition

# Continuations

- A programming technique for all forms of "non-local" control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls
- Essentially it's a higher-order function version of GOTO

# Continuations

- Idea: Use functions to represent the control flow of a program
- Method: Each procedure takes a function as an extra argument to which to pass its result; outer procedure “returns” no result
- Function receiving the result called a continuation
- Continuation acts as “accumulator” for work still to be done

# Continuation Passing Style

- Writing procedures such that all procedure calls take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)

# Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.
- A formalization of non-local control flow in denotational semantics
- Possible intermediate state in compiling functional code

# Why CPS?

- Makes order of evaluation explicitly clear
- Allocates variables (to become registers) for each step of computation
- Essentially converts functional programs into imperative ones
  - Major step for compiling to assembly or byte code
- Tail recursion easily identified
- Strict forward recursion converted to tail recursion
  - At the expense of building large closures in heap

# Other Uses for Continuations

- CPS designed to preserve order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

# Example

- Simple reporting continuation:

```
# let report x = (print_int x; print_newline( ) );;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```
# let addk (a, b) k = k (a + b);;
val addk : int * int -> (int -> 'a) -> 'a = <fun>
# addk (22, 20) report;;
2
- : unit = ()
```

# Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation
- Examples:

```
# let subk (x, y) k = k(x - y);;
val subk : int * int -> (int -> 'a) -> 'a = <fun>
# let eqk (x, y) k = k(x = y);;
val eqk : 'a * 'a -> (bool -> 'b) -> 'b = <fun>
# let timesk (x, y) k = k(x * y);;
val timesk : int * int -> (int -> 'a) -> 'a = <fun>
```

# Nesting Continuations

```
# let add_triple (x, y, z) = (x + y) + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple (x,y,z)=let p = x + y in p + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple_k (x, y, z) k =
   addk (x, y) (fun p -> addk (p, z) k );;
val add_triple_k: int * int * int -> (int -> 'a) ->
   'a = <fun>
```

# add_three: a different order

- # let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?

- let add_triple_k (x, y, z) k =

# add_three: a different order

- \# let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?

- let add_triple_k (x, y, z) k =

```
      addk (y,z) (fun r -> addk(x,r) k)
```

# Recursive Functions

- Recall:

```
# let rec factorial n =
    if n = 0 then 1 else n * factorial (n - 1);;
  val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

# Terms

- A function is in Direct Style when it returns its result back to the caller.
- A function is in Continuation Passing Style when it, and every function call in it, passes its result to another function.
- Instead of returning the result to the caller, we pass it forward to another function giving the computation after the call.

# Recursive Functions

```
# let rec factorial n =
    let b = (n = 0) in (* First computation *)
    if b then 1 (* Returned value *)
    else let s = n – 1 in (* Second computation *)
         let r = factorial s in  (* Third computation *)
         n * r (* Returned value *) ;;
val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

# Recursive Functions

```
# let rec factorialk n k =
   eqk (n, 0)
   (fun b ->  (* First computation *)
    if b then k 1 (* Passed value *)
    else subk (n, 1)  (* Second computation *)
    (fun s -> factorialk s  (* Third computation *)
      (fun r -> timesk (n, r) k))) (* Passed value *)
val factorialk : int -> (int -> ‘a) -> ‘a = <fun>
# factorialk 5 report;;
120
- : unit = ()
```

# Recursive Functions

- To make recursive call, must build intermediate continuation to
  - take recursive value: r
  - build it to final result: n * r
  - And pass it to final continuation:
  - `times (n, r) k = k (n * r)`

# Example: CPS for length

```
let rec length list = match list with [] -> 0
    | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?

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let rec length list = match list with [] -> 0
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```
let rec length list = match list with [] -> 0
    | (a :: bs) -> let r1 = length bs in 1 + r1
```

# Example: CPS for length

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#let rec length list = match list with [] -> 0
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# Example: CPS for length

```
#let rec length list = match list with [] -> 0
    | (a :: bs) -> let r1 = length bs in 1 + r1
```

What is the CSP version of this?

```
#let rec lengthk list k = match list with [ ] -> k 0
   | x :: xs -> lengthk xs (fun r -> addk (r,1) k);;
val lengthk : 'a list -> (int -> 'b) -> 'b = <fun>
# lengthk [2;4;6;8] report;;
4
- : unit = ()
```

# CPS for sum

```
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
```

# CPS for sum

```
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> let r1 = sum xs  in x + r1;;
```

# CPS for sum

```
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
     | x :: xs -> sumk xs  (fun r1 -> addk x r1 k);;
```

# CPS for sum

```
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
     | x :: xs -> sumk xs  (fun r1 -> addk (x, r1) k);;
val sumk : int list -> (int -> 'a) -> 'a = <fun>
# sumk [2;4;6;8] report;;
20
- : unit = ()
```

# CPS for Higher Order Functions

- In CPS, every procedure / function takes a continuation to receive its result
- Procedures passed as arguments take continuations
- Procedures returned as results take continuations
- CPS version of higher-order functions must expect input procedures to take continuations

# Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

# Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k =
```

# Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->     true
```

# Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
```

# Example: all

```
#let rec all (p, l) = match l with [] -> true
     | (x :: xs) -> let b = p x in
         if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) ->
```

# Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
```

# Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
        (fun b -> if b then                 else
    )
```

# Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
          (fun b -> if b then allk (pk, xs) k else k
false)
val allk : ('a -> (bool -> 'b) -> 'b) * 'a list ->
(bool -> 'b) -> 'b = <fun>
```

# Terminology: Review

- A function is in Direct Style when it returns its result back to the caller.
- A function is in Continuation Passing Style when it, and every function call in it, passes its result to another function.
- A Tail Call occurs when a function returns the result of another function call without any more computations (eg tail recursion)
- Instead of returning the result to the caller, we pass it forward to another function giving the computation after the call.

# CPS Transformation

- Step 1: Add continuation argument to any function definition:
  - let f arg = e $\Rightarrow$ let f arg k = e
  - Idea: Every function takes an extra parameter saying where the result goes
- Step 2: A simple expression in tail position should be passed to a continuation instead of returned:
  - return a $\Rightarrow$ k a
  - Assuming a is a constant or variable.
  - “Simple” = “No available function calls.”

# CPS Transformation

- Step 3: Pass the current continuation to every function call in tail position
  - return f arg $\Rightarrow$ f arg k
  - The function "isn't going to return," so we need to tell it where to put the result.

# CPS Transformation

- Step 4: Each function call not in tail position needs to be converted to take a new continuation (containing the old continuation as appropriate)
  - return op (f arg) $\Rightarrow$ f arg (fun r -> k(op r))
  - op represents a primitive operation

  - return  g(f arg) $\Rightarrow$ f arg (fun r-> g r k)

# Example

**Before:**

```
let rec add_list lst =
match lst with
  [ ] -> 0
| 0 :: xs -> add_list xs
| x :: xs -> (+) x
   (add_list xs);;
```

**After:**

```
let rec add_listk lst k =
                        (* rule 1 *)
match lst with
| [ ] -> k 0 (* rule 2 *)
| 0 :: xs -> add_listk xs k
                           (* rule 3 *)
| x :: xs -> add_listk xs
           (fun r -> k ((+) x r));;
                     (* rule 4 *)
```

# Other Uses for Continuations

- CPS designed to preserve  order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

# Exceptions - Example

```
# exception Zero;;
exception Zero
# let rec list_mult_aux list =
    match list with [ ] -> 1
    | x :: xs ->
     if x = 0 then raise Zero
                else x * list_mult_aux xs;;
val list_mult_aux : int list -> int = <fun>
```

# Exceptions - Example

```
# let list_mult list =
    try list_mult_aux list with Zero -> 0;;
val list_mult : int list -> int = <fun>
# list_mult [3;4;2];;
- : int = 24
# list_mult [7;4;0];;
- : int = 0
# list_mult_aux [7;4;0];;
Exception: Zero.
```

# Exceptions

- When an exception is raised
  - The current computation is aborted
  - Control is “thrown” back up the call stack until a matching handler is found
  - All the intermediate calls waiting for a return values are thrown away

# Implementing Exceptions

```
# let multkp (m, n) k =
  let r = m * n in
   (print_string "product result: ";
    print_int r; print_string "\n";
    k r);;
val multkp : int ( int -> (int -> 'a) -> 'a =
  <fun>
```

# Implementing Exceptions

```
# let rec list_multk_aux list k kexcp =
   match list with [ ] -> k 1
   | x :: xs -> if x = 0 then  kexcp  0
     else list_multk_aux xs
          (fun r -> multkp (x, r) k) kexcp;;
val list_multk_aux : int list -> (int -> 'a) -> (int -> 'a)
  -> 'a = <fun>
# let rec list_multk list k = list_multk_aux list  k  k;;
val list_multk : int list -> (int -> 'a) -> 'a = <fun>
```

# Implementing Exceptions

```
# list_multk [3;4;2] report;;
product result: 2
product result: 8
product result: 24
24
- : unit = ()
# list_multk [7;4;0] report;;
0
- : unit = ()
```