HW 11 – Lambda Calculus

CS 421 – Fall 2013

Revision 1.0

Assigned Thursday, November 21, 2011
Due Thursday, December 5, 2012, 19:59 pm
Extension 48 hours (20% penalty)

1 Change Log

1.1 Please use `hw11-solution.pdf` for the handin name.

1.0 Initial Release.

2 Turn-In Procedure

Answer the problems below, save your work as a PDF (either scanned if handwritten or converted from a program), and hand in the PDF. Your file should be named `hw11-solution.pdf`.

3 Objectives and Background

The purpose of this HW is to test your understanding of:

- Alpha and beta conversion in the lambda calculus
- The consequences of different evaluation schemes
- experience answering non-programming written questions similar to those on the final

4 Problems

1. (15 pts) Prove that $(\lambda y.xy)(\lambda x.\lambda y.yx)$ is α-equivalent $(\lambda z.xz)(\lambda y.\lambda x.xy)$

   You should label every use of α-conversion and congruence.

   Solution: By α-conversion

   $$\lambda y.xy \stackrel{\alpha}{\longrightarrow} \lambda z.xz.$$

   Because α-conversion implies α-equivalence, we have

   $$\lambda y.xy \quad \stackrel{\alpha}{\sim} \quad \lambda z.xz. \tag{1}$$

   Because α-equivalence is closed under application on the right, from **??** we have

   $$(\lambda y.xy)(\lambda x.\lambda y.yx) \quad \stackrel{\alpha}{\sim} \quad (\lambda z.xz)(\lambda x.\lambda y.yx). \tag{2}$$

   By α-conversion

   $$\lambda x.\lambda y.yx \stackrel{\alpha}{\longrightarrow} \lambda w.\lambda y.yw.$$

   Because α-conversion implies α-equivalence, we have

   $$\lambda x.\lambda y.yx \quad \stackrel{\alpha}{\sim} \quad \lambda w.\lambda y.yw. \tag{3}$$

By α-conversion

$$\lambda y.yw \xrightarrow{\alpha} \lambda x.xw.$$

Because α-conversion implies α-equivalence, we have

$$\lambda y.yw \overset{\alpha}{\sim} \lambda x.xw. \tag{4}$$

From **??**, by closure under abstractions of α-equivalence,

$$\lambda w.\lambda y.yw \overset{\alpha}{\sim} \lambda w.\lambda x.xw. \tag{5}$$

Because α-equivalence is closed under transitivity, from **??** and **??** we have

$$\lambda x.\lambda y.yx \overset{\alpha}{\sim} \lambda w.\lambda x.xw. \tag{6}$$

By α-conversion

$$\lambda w.\lambda x.xw \xrightarrow{\alpha} \lambda y.\lambda x.xy.$$

Because α-conversion implies α-equivalence, we have

$$\lambda w.\lambda x.xw \overset{\alpha}{\sim} \lambda y.\lambda x.xy. \tag{7}$$

Because α-equivalence is closed under transitivity, by **??** and **??** we have

$$\lambda x.\lambda y.yx \overset{\alpha}{\sim} \lambda y.\lambda x.xy. \tag{8}$$

Because α-equivalence is also closed under application on the left, from **??** we have

$$(\lambda z.xz)(\lambda x.\lambda y.yx) \overset{\alpha}{\sim} (\lambda z.xz)(\lambda y.\lambda x.xy). \tag{9}$$

Yet again, by transitivity of α-equivalence, from **??** and **??**, we have

$$(\lambda y.xy)(\lambda x.\lambda y.yx) \overset{\alpha}{\sim} (\lambda z.xz)(\lambda y.\lambda x.xy). \tag{10}$$

as was to be shown.

2. (15 pts) Given the following term:

$$(\lambda x.x(\lambda y.xy))((\lambda u.u)(\lambda w.w))$$

reduce this term as much as possible using each of

a. eager evaluation

b. lazy evaluation

c. unrestricted $\alpha\beta$-reduction (*i.e.* by $\alpha\beta$ conversion that can be applied anywhere)

Label each step of reduction with the rule justifying it. You do not need to label uses of congruence, or break them out as separate steps, in this problem.

Solution:

Eager Evaluation

$$\begin{aligned}
&(\lambda x.x(\lambda y.xy))((\lambda u.u)(\lambda w.w)) \\
&\xrightarrow{\beta}(\lambda x.x(\lambda y.xy))(\lambda w.w) \\
&\xrightarrow{\beta}(\lambda w.w)(\lambda y.(\lambda w.w)y) \\
&\xrightarrow{\beta}\lambda y.(\lambda w.w)y
\end{aligned}$$

(%x. x(%y.xy)) y

Lazy Evaluation

$$
\begin{aligned}
&(\lambda x.x(\lambda y.xy))((\lambda u.u)(\lambda w.w))\\
&\xrightarrow{\beta}((\lambda u.u)(\lambda w.w))(\lambda y.((\lambda u.u)(\lambda w.w))y)\\
&\xrightarrow{\beta}(\lambda w.w)(\lambda y.((\lambda u.u)(\lambda w.w))y)\\
&\xrightarrow{\beta}\lambda y.((\lambda u.u)(\lambda w.w))y
\end{aligned}
$$

Full $\alpha\beta$ Reduction: Continuing on from eager evaluation:

$$
\begin{aligned}
&\lambda y.(\lambda w.w)y\\
&\xrightarrow{\beta}\lambda y.y
\end{aligned}
$$

Full $\alpha\beta$ Reduction: Continuing on from lazy evaluation:

$$
\begin{aligned}
&\lambda y.((\lambda u.u)(\lambda w.w))y\\
&\xrightarrow{\beta}\lambda y.(\lambda w.w)y\\
&\xrightarrow{\beta}\lambda y.y
\end{aligned}
$$